A Lower Bound on Primorials and the Common Difference of Arithmetic Progression of Primes

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1 Abstract

Prime numbers in arithmetic progressions had been the subject of interest for many years. Here I derive a lower bound on the common difference of such an arithmetic progression. Then I give a lower bound on the product of primes i.e the primorials.

2 Notations

Suppose $2 = p_1, 3 = p_2, p_3, \ldots$ is the sequence of prime numbers. The primorial function is defined as $n\# = \prod_{p \le n} p$. The Chebyshev function is defined as $\theta(n) = \log(n\#)$. The prime counting function is $\pi(x) =$ number of primes $\le x$. All over here $\log n$ is the natural logarithm of n. And $\log^2 n$ is meant to be$= (\log n)^2$, not $\log\log n$.

3 The Results

Using elementary methods, Bonse proved[1] that

$p_1p_2 \ldots p_n > p_{n+1}^2 \forall n \ge 4$ and $p_1p_2 \ldots p_n > p_{n+1}^3 \forall n \ge 5$.

Without restriction of elementary methods, L. Pósa proved[2] that, given any $k > 1$, there exists an n_k such that,

$$p_1p_2 \ldots p_n > p_{n+1}^k \forall n \ge n_k.$$

In 2000, L.Panaitopol[3] gave another excellent bound,

$$p_1p_2 \ldots p_n > p_{n+1}^{n-\pi(n)} \forall n \ge 2.$$

I was actually working with the common difference of arithmetic progressions of primes. Suppose $p, p + d, p + 2d, \ldots, p + (n-1)d$ are n primes in A.P.$(n \ge 4)$. Using Panaitopol's inequality I arrived at the following lower bound on d : If $p_m \le n-1 < p_{m+1}$, then $d > n^{m-\pi(m)}$.

Then, modifying Panaipotol's proof , I arrived at this stronger bound on primorials :

Given any $k > 1$, there exists an n_k such that,$\forall n \ge n_k$,

$$p_1p_2 \ldots p_n > p_{n+1}^{n-\pi(n)+k}.$$

4 The Proofs

First let us prove the bound on the common difference d of an arithmetic progression of primes. Let $A_n = \{p, p+d, p+2d, \ldots, p+(n-1)d\}$ be an arithmetic progression of n primes.

We claim that $(n-1)\#|d$.

If this is not the case, then there exists prime $q < n$ which does not divide d. Consider $A_q = \{p, p+d, p+2d, \ldots, p+(q-1)d\}$. Now, if $\exists i, j\ (0 \leq i < j \leq q-1) : p+id \equiv p+jd \pmod q$, then we get $q|(j-i)d \Rightarrow q|j-i$ (since $q \not| d$). But this is impossible in light of $0 < j-i < q-1$. So the elements of A_q are congruent to distinct residues modulo q. Since A_q has q elements, so there exists some $p+kd \in A_q$ which is divisible by q. But, $p+kd \geq p \geq n > q$, [Note that $n \leq p$, otherwise $(p+pd) \in A_n$ but its not a prime.] hence $p+kd$ is composite, a contradiction.

Now, say $p_m \leq n-1 < p_{m+1}$. And $n \geq 4 \Rightarrow m \geq 2$. So using the aforesaid inequality of Panaitopol, we have

$d \geq (n-1)\# = p_m\# > p_{m+1}^{m-\pi(m)} \geq n^{m-\pi(m)}$(Proved).

Next we prove the lower bound on primorial function. First we prove the following lemma,

Lemma: For all $n \geq 59$, we have,

$$\log p_{n+1} < \log n + \log\log n + \frac{\log\log n - 0.4}{\log n}$$

Proof: We use a result due to Rosser and Schoenfeld[4],

$$p_n \leq n(\log n + \log\log n - 1/2) \qquad (\forall n \geq 20)\ldots(1)$$

From $\log(1+x) < x$ for $x > 0$, we get for $x = 1/n$ that,
$\log(n+1) < \log n + \frac{1}{n}$. We also get that,

$$\log\log(n+1) < \log(\log n + \frac{1}{n}) = \log\log n + \log(1 + \frac{1}{n\log n}) < \log\log n + \frac{1}{n\log n}.$$

Hence, along with (1), we obtain,
$\log p_{n+1} < \log(n+1) + \log(\log(n+1) + \log\log(n+1) - 1/2)$

$$\begin{aligned} &< \log n + \frac{1}{n} + \log\log n + \log(1 + \frac{1}{n\log n} + \frac{\log\log n}{\log n} + \frac{1}{n\log^2 n} - \frac{1}{2\log n}) \\ &< \log n + \frac{1}{n} + \log\log n + \frac{\log\log n}{\log n} + \frac{1}{n\log n} + \frac{1}{n\log^2 n} - \frac{1}{2\log n} \end{aligned}$$

So it remains to show that,

$$\frac{1}{n} + \frac{\log\log n}{\log n} + \frac{1}{n\log n} + \frac{1}{n\log^2 n} - \frac{1}{2\log n} < \frac{\log\log n - 0.4}{\log n}$$

or,

$$\frac{\log n}{n} + \log\log n + \frac{1}{n} + \frac{1}{n\log n} - \frac{1}{2} < \log\log n - 0.4$$

or,

$$\frac{\log n + 1}{n} + \frac{1}{n\log n} < 0.1$$

The LHS is a decreasing function of n, and at $n = 59(> e^4)$, it is < 0.1. So the above inequality holds $\forall n \geq 59$.

Next we use another estimate due to Rosser and Schoenfeld[4],

$$\pi(x) > \frac{x}{\log x} + \frac{x}{2\log^2 x} \qquad (\forall x \geq 59)\ldots(2)$$

and an estimate of $\theta(p_n)$ due to G.Robin[5],

$$\theta(p_n) > n(\log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n})(\forall n \geq 3)\ldots(3)$$

For $n \geq 59$, we use (2) and the aforesaid lemma to obtain,

$$n(1 - \frac{1}{\log n} - \frac{1}{2\log^2 n} + \frac{k}{n})(\log n + \log\log n + \frac{\log\log n - 0.4}{\log n})$$

$$> (n - \pi(n) + k)\log p_{n+1} \qquad \ldots(4)$$

Now, we wish to show that, given any k , $\theta(p_n) > (n - \pi(n) + k)\log p_{n+1}$ holds $\forall n \geq$ some n_k. So, in view of (3) and (4) , it suffices to show that,

$$(1 - \frac{1}{\log n} - \frac{1}{2\log^2 n} + \frac{k}{n})(\log n + \log\log n + \frac{\log\log n - 0.4}{\log n})$$

$$< (\log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n})$$

or,

$$\frac{k}{n}(\log n + \log\log n + \frac{\log\log n - 0.4}{\log n}) - \frac{\log\log n}{\log n} - \frac{\log\log n}{2\log^2 n}$$

$$-\frac{1}{2\log n} + \frac{\log\log n - 0.4}{\log n}(-\frac{1}{\log n} - \frac{1}{2\log^2 n}) - \frac{0.4}{\log n} < -\frac{2.1454}{\log n}$$

or,

$$\frac{1.2454}{\log n} + \frac{k}{n}\log\log n + \frac{k}{n}(\frac{\log\log n - 0.4}{\log n}) + \frac{k}{n}\log n < \frac{\log\log n}{\log n}$$

$$+\frac{0.5}{(\log n)^2}\log\log n + \frac{1}{\log n}(\frac{\log\log n - 0.4}{\log n}) + \frac{\log\log n - 0.4}{2(\log n)^3}$$

The last inequality holds for all n after some cut-off n_k. In fact, each of the following inequalities,

$$\frac{1.2454}{\log n} < \frac{\log\log n}{\log n} \quad , \quad \frac{k}{n}\log\log n < \frac{0.5}{(\log n)^2}\log\log n \quad ,$$

$$\frac{k}{n}(\frac{\log\log n - 0.4}{\log n}) < \frac{1}{\log n}(\frac{\log\log n - 0.4}{\log n}) \quad , \quad \frac{k}{n}\log n < \frac{\log\log n - 0.4}{2(\log n)^3},$$

holds after a certain [depending on k] cut-off n_k for n. Thus our required inequality also holds for all large enough n. Hence our proof is complete.

5 References

The references [1],[2],[4],[5] given below, are taken from [3].

[1] H.Rademacher, O.Toeplitz : *The enjoyment of mathematics.* Princeton Univ. Press, 1957.

[2] L.Pósa : *Über eine Eigenschaft der Primzahlen* (Hungarian). Mat. Lapok 11(1960), 124-129.

[3] L.Panaitopol: *An Inequality Involving Prime Numbers.* Univ. Beograd. Publ. Elektrotehn.Fak. Ser.Mat.11(2000), 33-35.

[4] J.B.Rosser, L.Schoenfeld : *Approximate formulas for some functions of prime numbers.* Illinois J.Math. 6(1962), 64-89.

[5] G.Robin : *Estimation de la fonction de Tschebyshev θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$, nombre des diviseurs premier de* n.Acta. Arith. 43(1983), 367-389.

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