Mock Interview Problems

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1 Problems

1.1 Number Theory and Combinatorics

1. Suppose $n > 2$ is a positive integer. Show that $2^n - 1$ can not be a power of 3.

2. A number n has distinct digits and the digits are in increasing order from left to right. Prove that the sum of the digits of the number $9n$ will always be 9.

3. Suppose $a_1, \cdots, a_n$ are numbers from the set $\{-1, 1\}$ such that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$. Show that n must be divisible by 4.

4. If for some positive integer n, the numbers 2^n and 5^n have same first digit d, then find d.

5. Find the number of 6-digit natural numbers which have no digit repeated, even digits occurring at even places, odd digits occurring at odd places and are divisible by 4.

6. You have to determine the number of ways can we fill up an $m \times n$ array of numbers with $(+1)$ or (-1) such that product of the numbers in each row and each column becomes (-1). Do this for (a) $m = 4, n = 5$; (b) $m = 4, n = 4$. Can you generalize?

7. Suppose $p_1, p_2, \cdots, p_{31}$ are distinct primes such that $p_1^4 + p_2^4 + \cdots + p_{31}^4$ is divisible by 30. Show that, there exists three consecutive primes among $p_1, p_2, \cdots, p_{31}$.

8. Find the greatest common divisor of the numbers
$$\binom{n}{r}, \binom{n+1}{r}, \cdots, \binom{n+r}{r}.$$

9. Determine the number of ways to select 3 distinct integers from $\{1, 2, \ldots, 30\}$ such that their sum is divisible by 3.

10. In how many ways can you colour 6 faces of a dice with 6 different colours, using each colour exactly once?

11. Ten points are selected on the positive x-axis, and five points are selected on the positive y-axis. The fifty segments connecting the ten selected points on x-axis to the five selected points on y-axis are drawn. What is the maximum possible number of points of intersection of these fifty segments that could lie in the interior of the first quadrant?

12. Suppose you go to a stationary shop to buy notebooks. If the shop sells notebooks of 5 different designs for the cover page and you want to buy 9 notebooks such that no design for the cover pages is missed (i.e. you want to pick at least one of each type), then find the number of ways to pick such a desired set of notebooks. Note, here ordering does not matter.

13. Suppose you went to a restaurant with five friends of yours. After having dinner, you decide to have ice-creams, one ice-cream per person. The menu card shows that the restaurant serves 8 flavours of ice-creams.

    (a) In how many ways can you select the six flavours to be ordered, such that each one of you have different flavour than others?

    (b) Suppose that the waiter comes and says that only 3 flavours (out of the 8) are available. Now, in how many ways can you select the six flavours to be ordered, such that no flavour is missed ?

    Note: In this problem, ordering of the flavours matters.

14. In how many ways can a row of 10 squares be each colored either red or green in such a way that no two red squares are adjacent?

15. A word consisting only of the letters A, B and C (some of these letters may not appear in the sequence) is called a *good* word if in that word, A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

16. Let $A = \{1, 2, \cdots, m\}$ and $B = \{1, 2, \cdots, n\}$.

    (a) Find the number of strictly increasing functions from A to B.

    (b) Find the number of non-decreasing functions from A to B.

    (c) Find the number of one-one functions from A to B.

    (d) Find the number of onto functions from A to B.

17. Find the number of functions f from $\{1, 2, \cdots, 15\}$ to itself such that $f(1) < f(3) < f(2)$.

18. Suppose $f : \mathbb{N} \to \mathbb{N}$ has the property that for every $n \in \mathbb{N}$,

$$f(1) + f(2) + \cdots + f(n) = c_n^3 \leq n^3$$

    where $c_n \in \mathbb{N}$. Find $f(n)$.

19. State Principle of Inclusion and Exclusion. Does it hold if we interchange the symbols $\cup$ and $\cap$?

20. A 10 digit number is called interesting if its digits are distinct and is divisible by 11111. Then find the number of interesting numbers.

1.2 Geometry

1. Let AD, BE, CF be the altitudes of $\triangle ABC$. Suppose P is a point inside the triangle. Let x, y, z be the perpendicular distances of P from the sides BC, CA, AB. Show that the value of

$$\frac{x}{AD} + \frac{y}{BE} + \frac{z}{CF}$$

   does not depend on P.

2. Let ABC be an acute-angled triangle, and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.

3. In a triangle ABC, the incircle touches the sides BC, CA, AB at D, E, F respectively. If the radius if the incircle is 4 units and if BD, CE, AF are consecutive integers, find the sides of the triangle ABC.

4. Let ABC be a triangle and consider two points D, E on BC such that $BD = EC$ and $\angle BAD = \angle EAC$. Show that $AB = AC$.

5. Segments AB and CD of length 1 intersect at O, such that $\angle AOC = 60°$. Prove that, $AC + BD \geq 1$.

6. Suppose $ABCD$ is a convex quadrilateral. Let E, F be points on AB such that $AE = EF = FB$. And let G, H be points on CD such that $CG = GH = HD$. If $ABCD$ has area 60 sq.unit, then find the area of $EFGH$.

1.3 Algebra

1. Suppose a, b, c are odd integers. Show that, the equation $ax^2 + bx + c = 0$ can not have any rational root.

2. Suppose a, b, c are non-zero real numbers with $a + b + c \neq 0$ and satisfying

$$a^{-1} + b^{-1} + c^{-1} = (a + b + c)^{-1}.$$

   Show that for any odd integer n,

$$(a + b + c)^n = a^n + b^n + c^n.$$

3. Suppose x, y, z are real numbers satisfying $x+y+z = 0$ and $xy+yz+zx = -3$. Show that the value of $x^3y + y^3z + z^3x$ is constant.

4. Suppose $\alpha \in \mathbb{C}$ satisfies $a+1/a+a^2+1/a^2+1=0$. Find $a^m+1/a^m+a^{2m}+1/a^{2m}$.

5. Define $\{x_n\}_{n\geq 1}$ by $x_1 = 1/2$ and $x_{n+1} = x_n^2 + x_n$ for $n \geq 1$. Find the greatest integer less than
$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_{2019}+1}.$$

6. Let $\{x_n\}_{n\geq 1}$ be a sequence satisfying $x_{n+1} = \dfrac{\sqrt{3}x_n - 1}{\sqrt{3} + x_n}$, $n \geq 1$. Prove that this sequence is periodic and also find the period.

7. For real numbers a, b, c define $s_n = a^n + b^n + c^n$. Suppose $s_1 = 2, s_2 = 6$ and $s_3 = 14$. Prove that $|s_n^2 - s_{n-1}s_{n+1}| = 8$ holds for all $n > 1$.

8. Find all values of $a \in \mathbb{R}$ such that the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has only real roots.

9. Find all positive real numbers x, y satisfying the system of equations:
$$\sqrt{x}\Big(1 + \frac{1}{x+y}\Big) = \frac{3}{2}, \quad \sqrt{y}\Big(1 - \frac{1}{x+y}\Big) = \frac{1}{2}.$$

10. Suppose that $z_1, \cdots, z_n$ and $w_1, \cdots, w_n$ are complex numbers with $|z_i| \leq 1$ and $|w_i| \leq 1$ for each $i = 1, 2, \cdots, n$. Show that,
$$|z_1 z_2 \cdots z_n - w_1 w_2 \cdots w_n| \leq |z_1 - w_1| + |z_2 - w_2| + \cdots + |z_n - w_n|.$$

1.4 Calculus

1. Evaluate the integral $\displaystyle\int_0^{\pi} \cos(x)\cos(2x)\cdots\cos(2017x)\, dx$.

2. Let $n \in \mathbb{N}$. Evaluate: $\displaystyle\lim_{x\to 0} \frac{1 - \cos(x)\cos(2x)\cdots\cos(nx)}{x^2}$.

3. Suppose a is a positive real number. Define a sequence $\{x_n\}_{n\geq 1}$ by
$$x_n = \frac{[a] + [2a] + \cdots + [na]}{n^2}, n \geq 1.$$
(Here $[t]$ denotes the greatest integer $\leq t$.) Prove that $\displaystyle\lim_{n\to\infty} x_n$ exists and also find the limit.

4. Evaluate the limit: $\displaystyle\lim_{n\to\infty} \frac{(1^4 + 2^4 + \cdots + n^4)^2}{(1^9 + 2^9 + \cdots + n^9)}$.

5. Evaluate the limit: $\displaystyle\lim_{n\to\infty} |\sin(\pi\sqrt{n^2+n+1})|$.

6. Suppose $f : [0, 2] \to \mathbb{R}$ is a continuous function such that $f(0) = f(2)$. Show that there exist $a, b \in [0, 2]$ such that $f(b) = f(a)$ and $b - a = 1$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x) - f(x/2) = x^2$ for every $x \in \mathbb{R}$. If $f(0) = 0$, then find $f(x)$.

8. Find, with proof, the value of $\lim_{n\to\infty} \tan^n \left(\frac{\pi}{4} + \frac{1}{n}\right)$.

9. Find a continuous function which is differentiable on $\mathbb{R}$ except at three points.

10. Find a function which is continuous only at $x = 0$.

11. Find a function which is continuous only at three points.

12. Find a function which is continuous at integers and discontinuous elsewhere.

13. Find a continuous function which is differentiable only at 0.

14. Find a continuous function which is differentiable only at three points.

15. Can you give an example of a function which is differentiable, but the derivative is not continuous?

16. Can you give an example of a function which is differentiable, the derivative is continuous but not differentiable?

17. Does there exist a continuous function f such that $f(x)$ is irrational if and only if x is rational?

18. Suppose f is a continuous function and $\{x_n\}_{n\geq 1}$ is a sequence of non-zero real numbers with $\lim_{n\to\infty} x_n = 0$ such that $f(x_n) = 0$ holds for each $n \geq 1$. Show that $f(0) = 0$. Is it necessary that $f'(0)$ exists and equals 0?

19. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If $\lim_{x\to 0} f'(x) = \ell$, is it necessary that $f'(0) = \ell$? (It is *not* given that $f'(x)$ is continuous at $x = 0$.)

20. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bijective function, which is differentiable at $x = 0$. Is it necessary that f^{-1} is differentiable at $f(0)$? If not, can you impose a (necessary and sufficient) condition for f^{-1} to be differentiable at $f(0)$?

21. Is it possible that an unbounded sequence has a convergent sub-sequence?

22. Suppose that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are two convergent sequences, with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Determine (with proof/counter-example) whether the following statements are true or false:

    (a) $\lim_{n\to\infty} (x_1 + \cdots + x_n) = \lim_{n\to\infty} (y_1 + \cdots + y_n)$.

    (b) $\lim_{n\to\infty} nx_n = \lim_{n\to\infty} ny_n$.

    (c) $\lim_{n\to\infty} (x_n)^n = \lim_{n\to\infty} (y_n)^n$.

    (Note, a statement is false if it fails to hold even for just one case.)

23. Suppose that x_n satisfies $x_{n+1} = \sqrt{6 + x_n}$ for every $n \geq 1$, and let $x_1 = \sqrt{6}$. Show that x_n converges and also find the limit.

24. Suppose x_n is a sequence such that $\lim_{n\to\infty} x_{2n} = L_1$, $\lim_{n\to\infty} x_{2n+1} = L_2$, and $\lim_{n\to\infty} x_{3n} = L_3$. Show that we must have $L_1 = L_2 = L_3$. Hence prove that $\lim_{n\to\infty} x_n$ exists and equals that common value L_1.

25. Suppose $\{x_n\}_{n\geq 1}$ is a sequence such that $0 < x_n < 1$ holds for each $n \geq 1$. Furthermore, suppose we have $4x_n(1 - x_{n+1}) > 1$ for every $n \geq 1$. Show that x_n converges and also find the limit.

26. Suppose $\{x_n\}_{n\geq 1}$ is a sequence of real numbers such that for every positive integer $k > 1$, $\lim_{n\to\infty} x_{kn} = 0$. Is it necessary that $\lim_{n\to\infty} x_n = 0$?

27. Let $x_0 = a, x_1 = b$ and define $x_{n+1} = \left(1 - \dfrac{1}{2n}\right)x_n + \dfrac{1}{2n}x_{n-1},\ \ n \geq 1$. Find $\lim_{n\to\infty} x_n$.

28. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, with the property that $\lim_{t\to\infty} \dfrac{f(2t)}{f(t)} = 1$. For any fixed natural number m, determine $\lim_{t\to\infty} \dfrac{f(mt)}{f(t)}$.

29. Suppose $f : (0, \infty) \to [0, \infty]$ is continuous and $\displaystyle\int_0^\infty f(x)dx < \infty$. Is it necessary that $\lim_{x\to\infty} f(x)$ exists?

30. Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is a convex function, with $\lim_{x\to 0} f(x) = 0$. Prove that $g(x) = f(x)/x$ (defined for $x > 0$) is increasing.

31. Carefully watch the following calculation.

$$\int_0^\pi \frac{2\ dx}{9\cos^2 x + 4\sin^2 x} = \int_0^\pi \frac{2\sec^2 x\ dx}{9 + 4\tan^2 x} = \frac{1}{3}\tan^{-1}\left(\frac{2\tan x}{3}\right)\Big|_{x=0}^{x=\pi}$$

Do you think the above is correct? If not, then state why and also provide a correct evaluation of the integral above.

32. Calculate $\displaystyle\int_0^1 x \ln x\ dx$. While doing by parts, why didn't you take an arbitrary constant in the indefinite integration inside the second integral?

33. Is it always true that $\displaystyle\int_a^b f'(x)dx = f(b)-f(a)$? Evaluate the integral $\displaystyle\int_{-1}^1 x^{-2}dx$.

34. Determine all continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy

$$\int_0^1 f(x)(x - f(x))dx = \frac{1}{12}.$$

35. Suppose $f(x)$ satisfies $f(x) + f(1-x) = 1$ for all $x \in [0,1]$. Let f^n denote f composed with itself n times. Find $\int_0^1 f^{2019}(x)dx$.

36. Let f be a real-valued continuous function which satisfies

$$f(\pi/6 + x) + f(\pi/3 - x) = \pi/2 \text{ for all } x \in \mathbb{R}.$$

Evaluate $\int_0^{\pi/2} \cos^2 f(x)dx$.

37. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that for every $x, y \in [0,1]$, $xf(y) + yf(x) \leq 1$ holds. Show that, $\int_0^1 f(x)dx \leq \frac{\pi}{4}$.
Can you tell any function for which equality holds here?

1.5 Miscellaneous

1. Evaluate $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3$.

2. Suppose $a > b$ and $c > d$. Does this imply $ac > bd$? Does this hold with the additional assumption that $b > 0, c > 0$ or $a > 0, d > 0$?

3. Find all pairs of positive integers (a, b) which satisfy $a^b = b^a$.

4. Roughly sketch the graphs of e^x, x^2 and $2x^2$ (in the same graph).

5. Roughly sketch the graphs of the following functions: (a) $x^2 + \frac{1}{x}$, (b) $x + \frac{1}{x^2}$, (c) $\frac{x-2}{(x-1)(x+3)}$, (d) $\frac{x-1}{(x-2)(x+3)}$, (e) $x \sin x$, (f) $(x^3 - 1)^{1/3}$, (g) $x^{1/x}$.

6. Suppose ABC is a triangle whose side lengths are integer and $\angle ABC = 90°$. Prove that in-radius of the triangle is also an integer.

7. Suppose to each point of a plane, we have assigned some real number in such a way that, sum of the numbers on the vertices of any square is zero. Prove that we have no other choice than assigning zero to every point.

8. Suppose we want to find $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$. We call this x and arrive at $x = \sqrt{2}^{\,x}$. Now, both $x = 2$ and $x = 4$ satisfy this equation. Which one is the correct value of x?

9. $f : A \to B$, $g : B \to C$ are two functions. $g \circ f : A \to C$ is a bijection. And $g : B \to C$ is also a one-one function. Then which of the following is necessarily true? (a) f is one-one, (b) g is onto, (c) f is onto.

10. Is it possible that $\log(x) = p(x)/q(x)$ holds for all $x > 0$, where $p(x), q(x)$ are polynomials?