An Identity Involving Bernoulli Numbers and a Proof of Euler's formula for $\zeta(2n)$

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Abstract

Let B_n be the sequence of Bernoulli numbers, with $B_1 = 1/2$. In this note we derive an identity involving Bernoulli numbers. Using that, we give a proof of Euler's formula for $\zeta(2n) = \sum_{k=1}^{\infty} k^{-2n}$, $n \geq 1$.

1. Introduction

Starting with $B_0 = 1$, we define Bernoulli numbers using the recursion

$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}, n \geq 1.$$

One can show that this definition is equivalent to define $\{B_n\}_{n\geq 0}$ as coefficients of the exponential generating function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{1}$$

In this note, we derive the following identity involving Bernoulli numbers

$$\sum_{k=0}^{n} \binom{2n+1}{2k} B_{2k} 2^{2k} = 2n+1, \ n \geq 0. \tag{2}$$

Using this, we shall prove Euler's formula for $\zeta(2n) = \sum_{k=1}^{\infty} k^{-2n}$, which states

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \ n \geq 1. \tag{3}$$

2. Deriving the identity involving Bernoulli numbers

For $n \geq 0$, define $a_n = B_n 2^n$ if n is even and $a_n = 0$ if n is odd. Set $b_n = 1$ for all $n \geq 0$. Consider the exponential generating functions

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \text{ and } U(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

Observe that, $B(x) = e^x$ and $A(x) = \sum_{n=0}^{\infty} B_{2n} \frac{(2x)^{2n}}{(2n)!} = \frac{1}{2}(U(2x) + U(-2x))$. Simplifying, we get $A(x) = x\frac{e^{2x}+1}{e^{2x}-1}$. Now, consider $C(x) = A(x)B(x)$. Writing $C(x) = \sum_{n=0}^{\infty} c_n x^n / n!$, we obtain

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \implies c_n = \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} B_{2l} 2^{2l}. \tag{4}$$

Next, observe that

$$C(x) - xe^x = A(x)e^x - xe^x = xe^x \left(\frac{e^{2x}+1}{e^{2x}-1} - 1 \right) = \frac{2x}{e^x - e^{-x}}.$$

Therefore, $C(x) - xe^x$ is an even function, which says that the coefficient of x^{2n+1} in it must be zero for all $n \geq 0$. Note that the coefficient of x^{2n+1} in $C(x) - xe^x$ is $c_{2n+1}/(2n+1)! - 1/(2n)!$. Hence, we get $c_{2n+1} = 2n+1$ for every $n \geq 0$. Now we use (4) to substitute for c_{2n+1} and what we get is precisely the identity (2).

3. Euler's formula for $\zeta(2n)$

Let us illustrate the main idea by proving $\zeta(2) = \frac{\pi^2}{6}$. We start with De Moivre's formula which states $(\cos\theta + i \sin\theta)^m = (\cos m\theta + i \sin m\theta)$. We rewrite it as $(\cot\theta + i)^m = \frac{\sin m\theta}{\sin^m \theta}(\cot m\theta + i)$, set $m = 2N+1$ and compare the imaginary parts on both sides to obtain

$$\binom{2N+1}{1}(\cot^2\theta)^N - \binom{2N+1}{3}(\cot^2\theta)^{N-1} + \cdots = \frac{\sin(2N+1)\theta}{(\sin\theta)^{2N+1}}.$$

Observe that for $\theta = \frac{k\pi}{2N+1}$, $(1 \leq k \leq N)$ the RHS vanishes. Hence we obtain that $\{\cot^2 \frac{k\pi}{2N+1} : k = 1, 2, \cdots, N\}$ are distinct roots of the polynomial

$$F(x) := \binom{2N+1}{1}x^N - \binom{2N+1}{3}x^{N-1} + \cdots = \sum_{k=0}^{N}(-1)^k\binom{2N+1}{2k+1}x^{N-k}.$$

The degree of $F(x)$ being N, those are the only roots. Therefore, using formula for sum of roots, we get

$$\sum_{k=1}^{N}\cot^2\frac{k\pi}{2N+1} = \frac{1}{2N+1}\binom{2N+1}{3} = \frac{N(2N-1)}{3}. \tag{5}$$

Now, we know that for $0 < x < \pi/2$, $\sin x < x < \tan x$ holds, which gives $\cot^2 x < 1/x^2 < \cot^2 x + 1$. And for each $1 \le k \le N$, we have $0 < \frac{k\pi}{2N+1} < \pi/2$. Therefore, we have

$$\cot^2\frac{k\pi}{2N+1} < \left(\frac{2N+1}{k\pi}\right)^2 < \cot^2\frac{k\pi}{2N+1} + 1,$$

for each $1 \le k \le N$. Summing up this for $k = 1, 2, \cdots, N$ and using (5), we get

$$\frac{N(2N-1)}{3} < \frac{(2N+1)^2}{\pi^2}\sum_{k=1}^{N}\frac{1}{k^2} < \frac{N(2N-1)}{3} + N. \tag{6}$$

Dividing both sides by $(2N+1)^2$ and letting $N \to \infty$, we see that both sides tend to $1/6$. Therefore, applying Sandwich theorem, we conclude that

$$\zeta(2) = \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Next, we move to the general formula for $\zeta(2n)$. We shall give similar bounds to $\sum_{k=1}^{N} 1/k^{2n}$ as we had in (6). For that we need a formula for $\sum_{k=1}^{N}\cot^{2n}\frac{k\pi}{2N+1}$. Recall that $\alpha_k := \cot^2\frac{k\pi}{2N+1}$ are the roots of $F(x)$. So we need a formula for sum of n-th powers of the roots of $F(x)$. The well-known Vieta's theorem provides a formula for $e_j :=$ sum of roots of $F(x)$ taken j at a time $(1 \le j \le N)$ which is given by

$$e_j = \frac{1}{2N+1}\binom{2N+1}{2j+1},\ 1 \le j \le N.$$

We set $e_0 = 1$. But what we need is a formula for $p_m = \sum_{k=1}^{N}\alpha_k^m$. Here an identity given by Newton comes to our rescue:

$$me_m = \sum_{k=1}^{m} (-1)^{k-1} e_{m-k} p_k. \tag{7}$$

Using the bounds $\cot^2 x < 1/x^2 < \cot^2 x + 1$, we get, for each $1 \le k \le N$,

$$\cot^{2n} \frac{k\pi}{2N+1} < \Big(\frac{2N+1}{k\pi}\Big)^{2n} < \Big(\cot^2 \frac{k\pi}{2N+1} + 1\Big)^n = \sum_{j=0}^{n} \binom{n}{j} \cot^{2j} \frac{k\pi}{2N+1}.$$

Summing up for $k = 1, 2, \cdots, N$ we obtain

$$p_n < \frac{(2N+1)^{2n}}{\pi^{2n}} \sum_{k=1}^{N} \frac{1}{k^{2n}} < \sum_{j=0}^{n} \binom{n}{j} p_j. \tag{8}$$

Note, here p_k's and e_k's are functions of N; we supressed N for simplicity in notation. It might be surprising for you that till now Bernoulli numbers have not come into the picture. It is the perfect time for their entry:

Lemma. For $n \ge 1$, we have

$$\lim_{N\to\infty} \frac{(-1)^{n-1} p_n}{N^{2n}} = \frac{B_{2n} 2^{4n-1}}{(2n)!} \tag{9}$$

Appealing to this, we can apply Sandwich theorem to conclude from (8) that

$$\zeta(2n) = \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{k^{2n}} = \Big(\frac{\pi}{2}\Big)^{2n} \lim_{N\to\infty} \frac{p_n}{N^{2n}} = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

So the proof for (3) will be completed once we prove the above lemma.

Proof of the Lemma: We shall use strong induction on n. The base case is easy to check. Let us assume it holds for $n = 1, 2, \cdots, (m-1)$, that is,

$$\lim_{N\to\infty} \frac{(-1)^{k-1} p_k}{N^{2k}} = \frac{B_{2k} 2^{4k-1}}{(2k)!}, \ 1 \le k < m.$$

We rewrite Newton's identity (7) as:

$$\frac{(-1)^{m-1} p_m}{N^{2m}} = \frac{me_m}{N^{2m}} - \sum_{k=1}^{m-1} \frac{e_{m-k}}{N^{2(m-k)}} \frac{(-1)^{k-1} p_k}{N^{2k}} \tag{*}$$

Observe that,

$$\lim_{N\to\infty} \frac{e_k}{N^{2k}} = \lim_{N\to\infty} \frac{1}{N^{2k}} \frac{1}{(2N+1)} \binom{2N+1}{2k+1} = \frac{2^{2k}}{(2k+1)!}.$$

Therefore, letting $N \to \infty$ in $(*)$ we obtain

$$\begin{aligned}
\lim_{N\to\infty} \frac{(-1)^{m-1} p_m}{N^{2m}} &= \frac{m 2^{2m}}{(2m+1)!} - \sum_{k=1}^{m-1} \frac{2^{2(m-k)}}{(2(m-k)+1)!} \frac{B_{2k} 2^{4k-1}}{(2k)!} \\
&= \frac{2^{2m}}{(2m+1)!} \left[m - \sum_{k=1}^{m-1} \binom{2m+1}{2k} B_{2k} 2^{2k-1} \right] \\
&= \frac{2^{2m}}{(2m+1)!} \left[m - \frac{1}{2} \left(2m+1-1-\binom{2m+1}{2m} B_{2m} 2^{2m} \right) \right] \\
&= \frac{B_{2m} 2^{4m-1}}{(2m)!}.
\end{aligned}$$

In the second last step, we have used the identity (2). This completes the induction and hence the proof of the lemma.