Some telescoping sums and products

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#1. Prove the identity $\prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{x}$.

#2. Simplify: $\cos\left(\frac{2\pi}{2^n-1}\right)\cos\left(\frac{4\pi}{2^n-1}\right)\cdots\cos\left(\frac{2^n\pi}{2^n-1}\right)$.

#3. Show that $\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$.

#4. Prove that $\frac{\sin x}{\cos x} + \frac{\sin(2x)}{\cos^2 x} + \cdots + \frac{\sin(nx)}{\cos^n x} = \cot x - \frac{\cos((n+1)x)}{\sin x \cos^n x}$.

#5. Evaluate the series: $\sum_{k=0}^{\infty} \cot^{-1}(k^2+k+1)$.

#6. Evaluate the series: $\sum_{k=1}^{\infty} \tan^{-1}\frac{1}{2k^2}$.

#7. Find a formula for $\sum_{k=1}^{n} \cos(kx)$. (Treat the case $x = 2m\pi, m \in \mathbb{Z}$ separately.)

#8. Simplify: $\frac{1}{\cos a - \cos 3a} + \frac{1}{\cos a - \cos 5a} + \cdots + \frac{1}{\cos a - \cos(2n+1)a}$.

#9. Evaluate the series: $\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{a}{2^n}$, assuming $a \neq k\pi$ for any $k \in \mathbb{Z}$.

#10. Show that $\sum_{n=1}^{\infty} 3^{n-1}\sin^3\frac{a}{3^n} = \frac{a - \sin a}{4}$.

#11. Simplify: $\sum_{k=1}^{n} \frac{1}{\sin(2^k x)}$, assuming that x is not of the form $k\pi/2^m$ for any $m \in \mathbb{Z}$.

#12. Prove that the average of the numbers $n \sin n^\circ$ where $n = 2, 4, 6 \ldots, 180$, is $\cot 1^\circ$.

#13. Evaluate: $(1 - \cot 1^\circ)(1 - \cot 2^\circ)\cdots(1 - \cot 44^\circ)$.

#14. Simplify: $\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \frac{\tan 4}{\cos 8} + \cdots + \frac{\tan 2^n}{\cos 2^{n+1}}$. (Here the angles are in radian.)

#15. Evaluate: $\prod_{k=1}^{n}\left(1-\tan^2\left(\frac{2^k\pi}{2^n+1}\right)\right)$.

#16. Show that $\left(\frac{1}{2}+\cos\frac{\pi}{20}\right)\left(\frac{1}{2}+\cos\frac{3\pi}{20}\right)\left(\frac{1}{2}+\cos\frac{9\pi}{20}\right)\left(\frac{1}{2}+\cos\frac{27\pi}{20}\right)=\frac{1}{16}$. Generalize.

#17. If x is not of the form $2^{k+1}(\pi/3+\ell\pi)$ for $k=1,2,\ldots,n$ and for any $\ell\in\mathbb{Z}$, show that

$$\prod_{k=1}^{n}\left(1-2\cos\frac{x}{2^k}\right)=(-1)^n\frac{1+2\cos x}{1+2\cos(x/2^n)}.$$

#18. Prove that $\prod_{k=1}^{n}\left(1+2\cos\left(\frac{3^k\cdot 2\pi}{3^n+1}\right)\right)=1$.