Telescoping Sums and Product in Trigonometry

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#1. $\displaystyle\prod_{n=1}^{\infty} \cos\frac{x}{2^n} = \frac{\sin x}{x}$

Proof $\displaystyle\prod_{n=1}^{N} \cos\frac{x}{2^n} = \frac{1}{2^N \sin\frac{x}{2^N}} \cdot 2^N \sin\frac{x}{2^N}\cos\frac{x}{2^N}\cos\frac{x}{2^{N-1}}\cdots\cos\frac{x}{2}$

$\displaystyle = \frac{\sin x}{2^N \sin\frac{x}{2^N}}$ $\quad\therefore\quad \displaystyle\prod_{n=1}^{\infty}\cos\frac{x}{2^n} = \lim_{N\to\infty}\prod_{n=1}^{N}\cos\frac{x}{2^n}$

$\displaystyle = \frac{\sin x}{x}\lim_{N\to\infty}\frac{x/2^N}{\sin x/2^N} = \frac{\sin x}{x}.$

#2 $\displaystyle\cos\frac{2\pi}{2^n-1}\cos\frac{4\pi}{2^n-1}\cdots\cos\frac{2^n\pi}{2^n-1} = \frac{1}{2^n}.$

Proof $\displaystyle \text{LHS} = \frac{1}{2^n\sin\frac{2\pi}{2^n-1}}\cdot 2^n \sin\frac{2\pi}{2^n-1}\cos\frac{2\pi}{2^n-1}\cos\frac{4\pi}{2^n-1}\cdots\cos\frac{2^n\pi}{2^n-1}$

$\displaystyle = \frac{1}{2^n\sin\frac{2\pi}{2^n-1}}\cdot\sin\frac{2^{n+1}\pi}{2^n-1} = \frac{\sin\left(2\pi+\frac{2\pi}{2^n-1}\right)}{2^n\sin\frac{2\pi}{2^n-1}} = \frac{1}{2^n}.$

* Doing #1 with telescoping Product:

$\displaystyle\prod_{n=1}^{\infty}\cos\frac{x}{2^n} = \lim_{N\to\infty}\prod_{n=1}^{N}\cos\frac{x}{2^n} = \lim_{N\to\infty}\prod_{n=1}^{N}\frac{\sin\frac{x}{2^{n-1}}}{2\sin\frac{x}{2^n}} = \lim_{N\to\infty}\frac{1}{2^N}\frac{\sin x}{\sin\frac{x}{2^N}}$

$\displaystyle = \frac{\sin x}{x}.$

#3. $\displaystyle\frac{1}{\cos 0^\circ\cos 1^\circ} + \frac{1}{\cos 1^\circ\cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ\cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$

Proof $\displaystyle\frac{1}{\cos K^\circ\cos(K+1)^\circ} = \frac{1}{\sin 1^\circ}\,\frac{\sin((K+1)^\circ-K^\circ)}{\cos K^\circ\cos(K+1)^\circ} = \frac{1}{\sin 1^\circ}\left[\tan(K+1)^\circ - \tan K^\circ\right]$

$\displaystyle\therefore\ \sum_{K=0}^{88}\frac{1}{\cos K^\circ\cos(K+1)^\circ} = \sum_{K=0}^{88}\frac{1}{\sin 1^\circ}\left[\tan(K+1)^\circ - \tan K^\circ\right]$

$\displaystyle = \frac{1}{\sin 1^\circ}(\tan 89^\circ - \tan 0^\circ) = \frac{\cot 1^\circ}{\sin 1^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ},$

#4. $\dfrac{\sin x}{\cos x} + \dfrac{\sin 2x}{\cos^2 x} + \dots + \dfrac{\sin nx}{\cos^n x} = \cot x - \dfrac{\cos(n+1)x}{\sin x \cos^n x}$

Proof. $\dfrac{\sin kx}{\cos^k x} = \dfrac{\sin x \sin kx}{\sin x \cos^k x} = \dfrac{\cos x \cos kx - (\cos x \cos kx - \sin x \sin kx)}{\sin x \cos^k x}$

$\therefore \boxed{\dfrac{\sin kx}{\cos^k x} = \dfrac{\cos kx}{\sin x \cos^{k-1} x} - \dfrac{\cos(k+1)x}{\sin x \cos^k x}}$

$\Rightarrow \sum_{k=1}^{n} \dfrac{\sin kx}{\cos^k x} = \dfrac{\cos x}{\sin x . \cos^{1-1} x} - \dfrac{\cos(n+1)x}{\sin x \cos^n x}$

$= \cot x - \dfrac{\cos(n+1)x}{\sin x \cos^n x}.$

5. $\sum_{k=0}^{n} \tan^{-1} \dfrac{1}{k^2+k+1}$.

Sol. $\left(\sum_{k=0}^{n} \cot^{-1}(k^2+k+1) =\right) \sum_{k=0}^{n} \tan^{-1} \dfrac{1}{k^2+k+1} = \sum_{k=0}^{n} \tan^{-1} \dfrac{(k+1)-k}{1+(k+1)k}$.

$= \sum_{k=0}^{n} [\tan^{-1}(k+1) - \tan^{-1} k] = \tan^{-1}(n+1) - \tan^{-1} 0$

$= \tan^{-1}(n+1)$

special case: $\sum_{k=0}^{\infty} \cot^{-1}(k^2+k+1) = \lim_{n\to\infty} \cot^{-1}\left(\dfrac{1}{n+1}\right)$

$= \cot^{-1} 0 = \dfrac{\pi}{2}$

6. $\sum_{k=1}^{n} \tan^{-1} \dfrac{1}{2k^2} = \tan^{-1} \dfrac{n}{n+1}$

Proof. $\tan^{-1} \dfrac{k}{k+1} - \tan^{-1} \dfrac{k-1}{k}$

$= \tan^{-1} \dfrac{\dfrac{k}{k+1} - \dfrac{k-1}{k}}{1 + \dfrac{k-1}{k+1}} \quad [\because k \neq 0]$

$= \tan^{-1} \dfrac{k^2-(k^2-1)}{k(k+1+k-1)} = \tan^{-1} \dfrac{1}{2k^2}$

$\therefore \sum_{k=1}^{n} \tan^{-1} \dfrac{1}{2k^2} \left(= \sum_{k=1}^{n} \cot^{-1} 2k^2\right) = \sum_{k=1}^{n} \left(\tan^{-1} \dfrac{k}{k+1} - \tan^{-1} \dfrac{k-1}{k}\right)$

$= \tan^{-1} \dfrac{n}{n+1} - \tan^{-1} \dfrac{0}{1} = \tan^{-1} \dfrac{n}{n+1}$.

Special case. $\sum_{k=1}^{\infty} \cot^{-1} 2k^2 = \lim_{n\to\infty} \tan^{-1} \dfrac{n}{n+1} = \tan^{-1} 1 = \dfrac{\pi}{4}$,

\# 7. $\sum_{k=1}^{n} \cos kx$.

Sol If $x = 2m\pi$ $(m \in \mathbb{Z})$, $\sum_{k=1}^{n} \cos kx = \sum_{k=1}^{n} 1 = n$.

Now we assume $x \neq 2m\pi$ and hence we can divide by or cancel out $\sin x/2$.

$$\sum_{k=1}^{n} \cos kx = \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin\frac{x}{2}\cos kx$$

$$= \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} \left[\sin\left(k+\tfrac{1}{2}\right)x - \sin\left(k-\tfrac{1}{2}\right)x\right]$$

$$= \frac{1}{2\sin\frac{x}{2}}\left[\sin\left(n+\tfrac{1}{2}\right)x - \sin \tfrac{1}{2}x\right] = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \tfrac{1}{2}.$$

\# 8. $\frac{1}{\cos a - \cos 3a} + \frac{1}{\cos a - \cos 5a} + \cdots + \frac{1}{\cos a - \cos(2n+1)a} = ?$

$[n \in \mathbb{Z}^+,\ a \in \mathbb{R} : \frac{a}{\pi} \in \mathbb{R}\setminus\mathbb{Q}]$

Sol

$$\sum_{k=1}^{n} \frac{1}{\cos a - \cos(2k+1)a} = \frac{1}{\sin a}\sum_{k=1}^{n} \frac{\sin((k+1)a - ka)}{2\sin(k+1)a\,\sin ka}$$

$$= \frac{1}{2\sin a}\sum_{k=1}^{n}\left[\cot ka - \cot(k+1)a\right] = \frac{1}{2\sin a}\left[\cot a - \cot(n+1)a\right]$$

\# 9. $\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{a}{2^n}$. $[a \neq k\pi, (k \in \mathbb{Z})]$

Sol $\cot 2x = \frac{\cot^2 x - 1}{2\cot x} \Rightarrow \boxed{2\cot 2x = \cot x - \tan x}$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{a}{2^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n}\left[\cot\frac{a}{2^n} - 2\cot\frac{a}{2^{n-1}}\right]$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{2^n}\cot\frac{a}{2^n} - \frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}}\right) = \lim_{N\to\infty}\left[\frac{1}{2^N}\cot\frac{a}{2^N} - \cot a\right]$$

$$= \frac{1}{a}\left(\lim_{N\to\infty}\frac{a/2^N}{\sin\frac{a}{2^N}}\right)\left(\lim_{N\to\infty}\cos\frac{a}{2^N}\right) - \cot a = \frac{1}{a}\cdot 1\cdot \cos 0 - \cot a$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{a}{2^n} = \frac{1}{a} - \frac{1}{\tan a}.$$

#10. $\sum_{n=1}^{\infty} 3^{n-1} \sin^3 \frac{a}{3^n} = \frac{1}{4}(a - \sin a)$

Proof

$\sin 3x = 3\sin x - 4\sin^3 x \quad \therefore \quad \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$$\therefore \sum_{n=1}^{\infty} 3^{n-1} \sin^3 \frac{a}{3^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \left[\frac{3^{n-1}}{4}\left(3\sin\frac{a}{3^n} - \sin\frac{a}{3^{n-1}}\right)\right]$$

$$= \frac{1}{4} \lim_{N\to\infty} \sum_{n=1}^{N} \left(3^n \sin\frac{a}{3^n} - 3^{n-1}\sin\frac{a}{3^{n-1}}\right) = \frac{1}{4}\lim_{N\to\infty}\left(3^N \sin\frac{a}{3^N} - \sin a\right)$$

$$= \frac{a}{4}\left(\lim_{N\to\infty} \frac{\sin a/3^N}{a/3^N}\right) - \frac{1}{4}\sin a = \frac{1}{4}(a - \sin a)$$

#11. $\sum_{k=1}^{n} \frac{1}{\sin 2^k x} = \cot x - \cot 2^n x$. [$n \in \mathbb{Z}^+$, $x \in \mathbb{R}$ provided, $x \neq \frac{k\pi}{2^m}$ where $k \in \mathbb{Z}$, $m = 0, 1, \ldots, n$]

Proof

$$\cot 2^{k-1}x - \cot 2^k x$$

$$= \frac{\cos 2^{k-1}x}{\sin 2^{k-1}x} - \frac{\cos 2^k x}{\sin 2^k x} = \frac{\sin(2^k x - 2^{k-1}x)}{\sin 2^{k-1}x \sin 2^k x} = \frac{1}{\sin 2^k x}$$

$$\therefore \boxed{\frac{1}{\sin 2^k x} = \cot 2^{k-1}x - \cot 2^k x} \ (*)$$

$$\therefore \sum_{k=1}^{n} \frac{1}{\sin 2^k x} = \cot x - \cot 2^n x.$$

(*) Such identities are not randomly derived or ~~guessed~~ guessed. You can just plug in small values ~~of~~ n to get the correct identity. #4 is also an example, you can plug in $n=1$, $n=2$ and can guess which ~~should be~~ way we should move.

#12. Prove that, average of $n \sin n^\circ$, $n = 2, 4, 6, \ldots, 180$ is $\cot 1^\circ$.

Proof We need to show that, $\sum_{k=1}^{90} (2k)\sin(2k)^\circ = 90\cot 1^\circ$.

Now, $2k\sin(2k)^\circ \sin 1^\circ = k[\cos(2k-1)^\circ - \cos(2k+1)^\circ]$

$$\therefore \sum_{k=1}^{90} 2k\sin(2k^\circ)\sin 1^\circ = \cos 1^\circ - \cos 3^\circ + 2\cos 3^\circ - 2\cos 5^\circ + 3\cos 5^\circ - \cdots + 90\cos 179^\circ - 90\cos 181^\circ$$

$$= \cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cdots + \cos 179^\circ + 90\cos 1^\circ$$

$= \cos 1^\circ + \cos 3^\circ + \dots + \cos 89^\circ + \cos 91^\circ + \dots + \cos 179^\circ + 90\cos 1^\circ$

$= \cos 1^\circ + \cos 3^\circ + \dots + \cos 89^\circ - \cos 89^\circ - \dots - \cos 1^\circ + 90\cos 1^\circ$

$= 90\cos 1^\circ \quad \therefore \sum_{k=1}^{90} 2k \sin(2k)^\circ = 90 \cot 1^\circ$

13. $(1-\cot 1^\circ)(1-\cot 2^\circ)\dots(1-\cot 44^\circ) = ?$

Sol $\quad 1 - \cot x^\circ = \dfrac{\sin x^\circ - \cos x^\circ}{\sin x^\circ} = \sqrt{2}\,\dfrac{\sin(x^\circ - 45^\circ)}{\sin x^\circ}$

$\therefore \prod_{x=1}^{44}(1-\cot x^\circ) = 2^{22}\,\dfrac{\sin(-44^\circ)}{\sin 1^\circ}\cdot\dfrac{\sin(-43^\circ)}{\sin 2^\circ}\dots\dfrac{\sin(-1^\circ)}{\sin 44^\circ}$

$= 2^{22}$ [there are 44 "$-$" signs]

14. $\dfrac{\tan 1}{\cos 2} + \dfrac{\tan 2}{\cos 4} + \dots + \dfrac{\tan 2^n}{\cos 2^{n+1}} = ?$

$n=0 \quad \dfrac{\tan 1}{\cos 2}$

$n=1 \quad \dfrac{\tan 1}{\cos 2} + \dfrac{\tan 2}{\cos 4}$

$\dfrac{\sin(2^{k+1}-2^k)}{\cos 2^k \cos 2^{k+1}} = \dfrac{\sin 1\cos 4 + \sin 2\cos 1}{\cos 1\cos 2\cos 4}$

$= \dfrac{\sin 1(\cos 4 + 2\cos^2 1)}{\cos 1\cos 2\cos 4}$

Sol $\quad \dfrac{\tan 2^k}{\cos 2^{k+1}} = \dfrac{\sin 2^k}{\cos 2^k \cos 2^{k+1}}$

$= \dfrac{\sin(2^{k+1}-2^k)}{\cos 2^k \cos 2^{k+1}} = \tan 2^{k+1} - \tan 2^k$

$\therefore \sum_{k=0}^{n} \dfrac{\tan 2^k}{\cos 2^{k+1}} = \tan 2^{n+1} - \tan 1$

$1 - t^2 \quad \dfrac{2^k\pi}{2^n+1}$

$= \dfrac{\cos \frac{2^{k+1}\pi}{2^n+1}}{\cos^2 \frac{2^k\pi}{2^n+1}}$

*Even More useful identity is: $\dfrac{\tan a}{\cos 2a} = \tan 2a - \tan a$.

15. $\prod_{k=1}^{n}\left\{1 - \tan^2\left(\dfrac{2^k\pi}{2^n+1}\right)\right\} = ?$

Sol $\quad \prod_{k=1}^{n}\left(1 - \tan^2 \dfrac{2^k\pi}{2^n+1}\right) = \prod_{k=1}^{n} \dfrac{\cos^2\frac{2^k\pi}{2^n+1} - \sin^2\frac{2^k\pi}{2^n+1}}{\cos^2\frac{2^k\pi}{2^n+1}}$

$= \prod_{k=1}^{n}\dfrac{\cos\frac{2^{k+1}\pi}{2^n+1}}{\cos\frac{2^k\pi}{2^n+1}} \cdot \prod_{k=1}^{n}\dfrac{1}{\cos\frac{2^k\pi}{2^n+1}} = \dfrac{\cos\frac{2^{n+1}\pi}{2^n+1}\cdot(-2^n)}{\cos\frac{2\pi}{2^n+1}}$

Here I used the identity written in the next page and several times before.

* $\cos\theta \cos 2\theta \cos 2^2\theta \cdots \cos 2^m\theta$

$$= \frac{1}{2^{m+1}} \cdot \frac{\sin 2^{m+1}\theta}{\sin\theta} \quad [\sin\theta \neq 0]$$

$2(2^n+1)-2$

$$\left[\therefore \prod_{k=1}^{n} \cos\frac{2^k\pi}{2^n+1} = \frac{1}{2^n}\,\frac{\sin\left(\frac{2^{n+1}\pi}{2^n+1}\right)}{\sin\left(\frac{2\pi}{2^n+1}\right)} = \frac{1}{2^n}\,\frac{\sin\left(2\pi - \frac{2\pi}{2^n+1}\right)}{\sin\frac{2\pi}{2^n+1}} = -1/2^n\right]$$

$$\therefore \prod_{k=1}^{n}\left(1-\tan^2\frac{2^k\pi}{2^n+1}\right) = (-2^n)\,\frac{\cos\left(2\pi-\frac{2\pi}{2^n+1}\right)}{\cos\frac{2\pi}{2^n+1}} = (-2^n)$$

(Ans)

Alt. (Better)

$$1-\tan^2 x = \frac{2\tan x}{\tan 2x}$$

$$\therefore \prod_{k=1}^{n}\left(1-\tan^2\frac{2^k\pi}{2^n+1}\right) = \prod_{k=1}^{n}\frac{2\tan\frac{2^k\pi}{2^n+1}}{\tan\frac{2^{k+1}\pi}{2^n+1}} = 2^n\,\frac{\tan\frac{2\pi}{2^n+1}}{\tan\frac{2^{n+1}\pi}{2^n+1}}$$

$$= 2^n\,\frac{\tan\frac{2\pi}{2^n+1}}{\tan\left(2\pi-\frac{2\pi}{2^n+1}\right)} = (-2^n).$$

#16. $\left(\frac12+\cos\frac{\pi}{20}\right)\left(\frac12+\cos\frac{3\pi}{20}\right)\left(\frac12+\cos\frac{9\pi}{20}\right)\left(\frac12+\cos\frac{27\pi}{20}\right)$

$1+2\cos x$

$\frac{\cos 3x/2}{\cos x/2} = 4\cos^2 x/2 - 3 = 2\cos x - 1$

$1+2\cos x = \frac{\cancel{4\cos^2 x - 2 + 1}}{2\cos x - 1}$

$= -2\cos 2x - 1$

$$\frac{\cos 3x}{\cos x} = 4\cos^2 x - 3 = 2\cos 2x - 1$$

$$\therefore 1+2\cos x = -(2\cos(x+\pi)-1)$$

$$= -\frac{\cos 3\left(\frac{x}{2}+\frac{\pi}{2}\right)}{\cos\left(\frac{x}{2}+\frac{\pi}{2}\right)} = \frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}$$

$$\therefore \prod_{k=0}^{3}\left(\frac12+\cos\frac{3^k\pi}{20}\right) = \frac{1}{16}\prod_{k=0}^{3}\frac{\sin\frac{3^{k+1}\pi}{40}}{\sin\frac{3^k\pi}{40}} = \frac{1}{16}\,\frac{\sin\frac{81\pi}{40}}{\sin\frac{\pi}{40}} = \frac{1}{16}.$$

Generalisation: $\prod_{k=0}^{n}\left(\frac12+\cos 3^k a\right) = \frac{1}{2^{n+1}}\cdot\frac{\sin 3^{n+1}a}{\sin a}$

In particular, if $a = \frac{m\pi}{3^{n+1}+(-1)^{m+1}}$, $(n\in\mathbb{N}, m\in\mathbb{Z})$ $\prod_{k=0}^{n}\left(\frac12+\cos 3^k a\right) = \frac{1}{2^{n+1}}$

Here we were given the case $n=3, m=4$.

$$\boxed{\text{If } a = \frac{m\pi}{3^n-(-1)^m},\ \prod_{k=0}^{n-1}\left(\frac12+\cos 3^k a\right) = \frac{1}{2^n} \quad \begin{array}{l} m\in\mathbb{Z} \\ n\in\mathbb{Z}^+ \end{array}}$$

17. $n \in \mathbb{Z}^+$, $x \in \mathbb{R}$: $x \neq 2^{k+1}\left(\frac{\pi}{3} + l\pi\right)$, $k = 1, 2, \ldots, n$; $l \in \mathbb{Z}$. Find–

$$\prod_{k=1}^{n} \left(1 - 2\cos\frac{x}{2^k}\right).$$

18. Prove that, $\displaystyle\prod_{k=1}^{n} \left(1 + 2\cos\frac{2\pi \cdot 3^k}{3^n + 1}\right) = 1.$

#17:

$$(1 - 2\cos\theta) = \frac{1 - 4\cos^2\theta}{1 + 2\cos\theta} = -\frac{1 + 2\cos 2\theta}{1 + 2\cos\theta}.$$

$$\therefore \prod_{k=1}^{n} \left(1 - 2\cos\frac{x}{2^k}\right) = \prod_{k=1}^{n} \left\{-\frac{\left(1 + 2\cos\frac{x}{2^{k-1}}\right)}{\left(1 + 2\cos\frac{x}{2^k}\right)}\right\} = (-1)^n \frac{1 + 2\cos x}{1 + 2\cos\frac{x}{2^n}}.$$

#18:

$$1 + 2\cos 2\theta = 3 - 4\sin^2\theta = \frac{\sin 3\theta}{\sin\theta}.$$

$$\therefore \prod_{k=1}^{n} \left\{1 + 2\cos 2\left(\frac{3^k\pi}{3^n+1}\right)\right\} = \prod_{k=1}^{n} \left(\frac{\sin\frac{3^{k+1}\pi}{3^n+1}}{\sin\frac{3^k\pi}{3^n+1}}\right) = \frac{\sin\frac{3^{n+1}\pi}{3^n+1}}{\sin\frac{3\pi}{3^n+1}} = 1.$$

$$\left[\because \sin\frac{3^{n+1}\pi}{3^n+1} = \sin\left(3\pi - \frac{3\pi}{3^n+1}\right)\right]$$