Complex Numbers (Introduction)

$\mathbb{C} = \{a + ib,\ a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$.

"Assume that all the laws of algebra (namely, commutativity, associativity etc.) also holds for numbers in $\mathbb{C}$"

$\mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}$

Define addition and multiplication as follows.

$$(a, b) + (c, d) = (a + c,\ b + d),$$

$$(a, b) \times (c, d) = (ac - bd,\ ad + bc).$$

Historical Introduction

Tartaglia

Can you tell a number such that p times that number is added to the cube of the number, then you get q?

Sol: Solve the eqn $x^3 + px = q$.

Formula Tartaglia had (which was a secret):

$$\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Cardano A solution of $x^3 = px + q$ is given by

$$\sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

<u>Cardano</u>: Divide 10 into two parts whose product is 40.

<u>Ans</u>: $5+\sqrt{-15}$, $5-\sqrt{-15}$.

<u>Others</u>: We don't believe!

<u>Cardano</u>: $5+\sqrt{-15}+5-\sqrt{-15}=10$

$$(5+\sqrt{-15})(5-\sqrt{-15}) = 5^2-(-15) = 40.$$

<u>Bombelli</u>: $x^3 = 15x+4$.

<u>Common man's solution</u>: $x=4$.

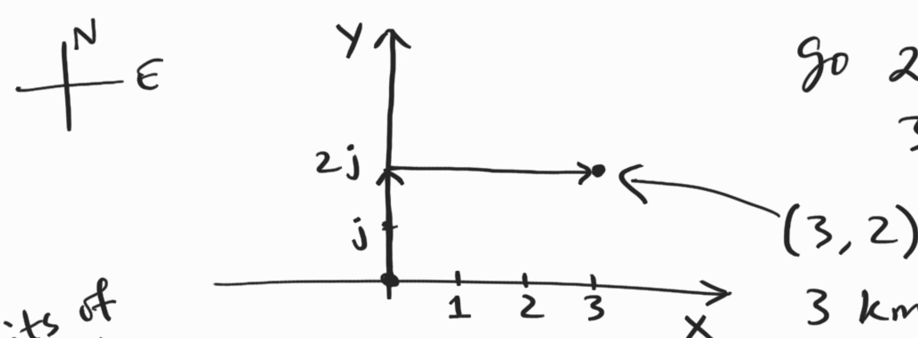
<u>Cardano's solution</u>: $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$.

$= 4$.

$$\left.\begin{aligned} \sqrt[3]{2+\sqrt{-121}} &= 2+\sqrt{-1} \\ \sqrt[3]{2-\sqrt{-121}} &= 2-\sqrt{-1} \end{aligned}\right\}$$

<u>Geometric Introduction</u>

Go 2 km N then 3 km E.



3 km E, 2 km N

$3\hat{i}+2\hat{j}$

$3+2j$

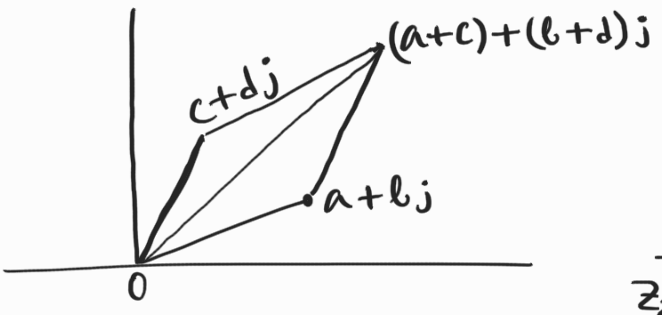
<u>units of the axes</u>

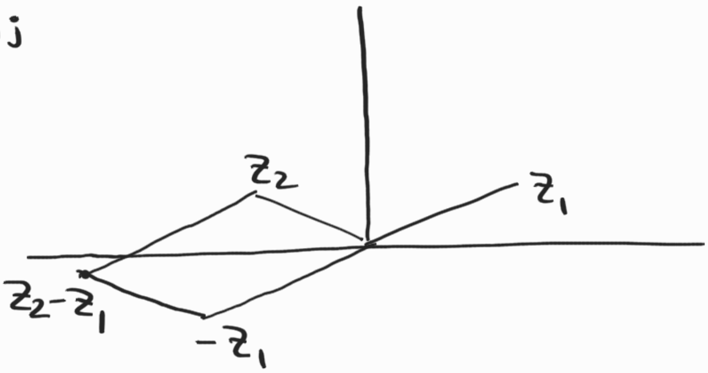
1 km E $\hookrightarrow$ 1   1 km N $\hookrightarrow$ j

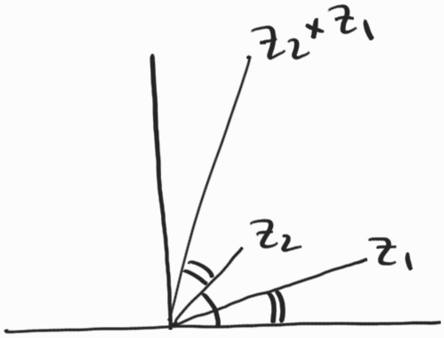
$z+(3+2j)$ <u>meaning</u>?

$$(a+bj)+(3+2j) = (a+3)+(b+2)j$$

a km E, b km N + 3 km E, 2 km N = (a+3) km E, (b+2) km N.

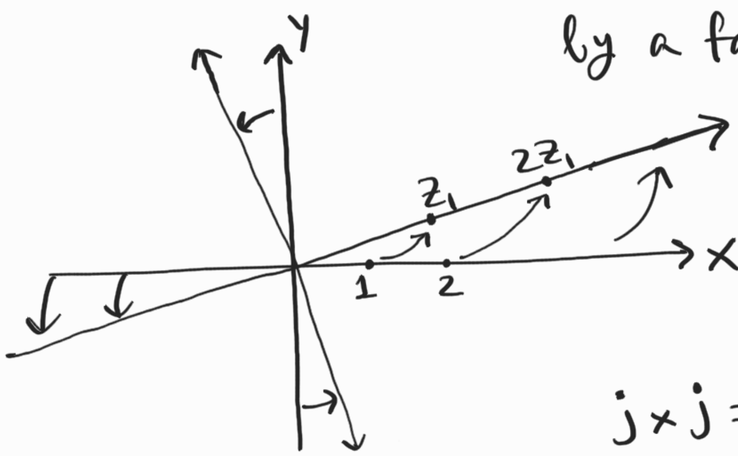




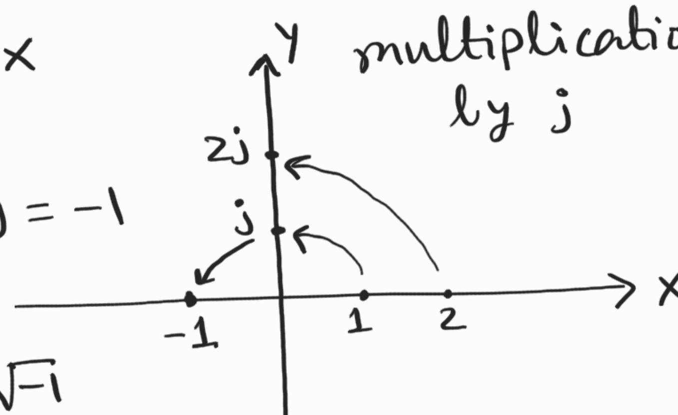


How to get $z_2 \times z_1$:

First rotate z_2 by an angle which same as the angle z_1 makes with the positive x-axis, and stretch by a factor $= |z_1| =$ length of z_1 from the origin.



multiplication by j



$j \times j = -1$

$j = \sqrt{-1}$

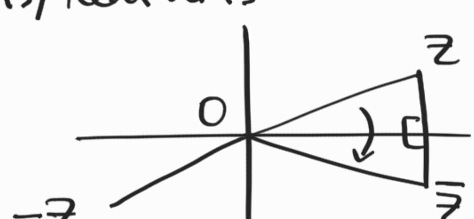
So, from now on, we will use $i = \sqrt{-1}$, instead of j.

$|z| =$ length of z from the origin

$\arg(z) =$ angle z makes with positive x-axis $\pm$ something ← We'll discuss later.

$\bar{z}$ = reflection of z in the x-axis/real axis

$$\boxed{z\,\bar{z} = |z|^2}$$

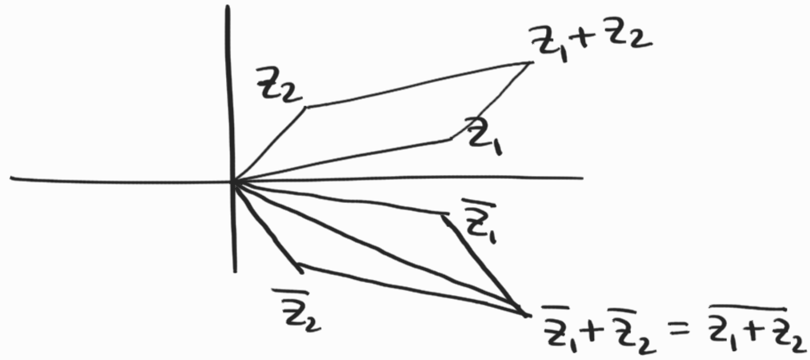


$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

$$\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$$

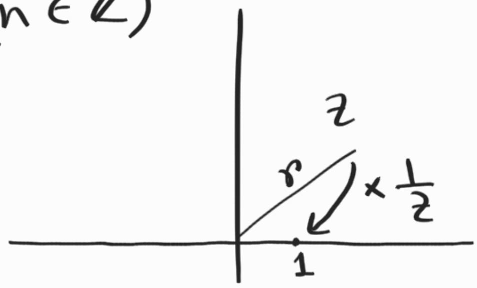
$$\overline{z^n} = \overline{z}^{\,n} \quad (n \in \mathbb{Z})$$



$$|z_1 z_2| = |z_1||z_2|$$

$$|z^n| = |z|^n$$

$$|z_1/z_2| = |z_1|/|z_2|$$



$$z = r(\cos\theta + i\sin\theta)$$

$$\Downarrow$$

$$\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$$

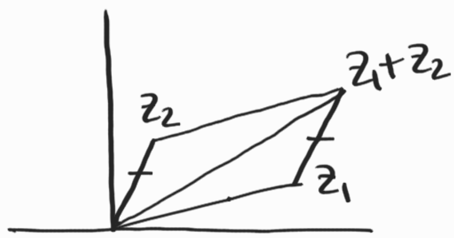
$$= \overline{z}/|z|^2$$

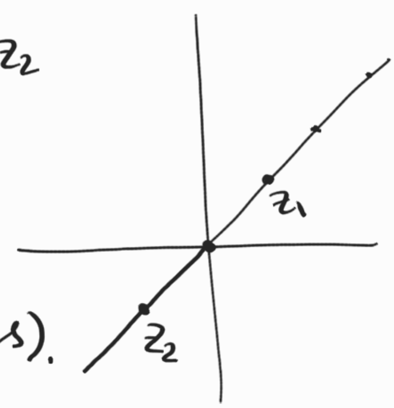
$$|z_1 + z_2| \le |z_1| + |z_2|,$$

with equality iff

$0, z_1, z_2$ or $0, z_2, z_1$

are collinear (in one of these orderings).

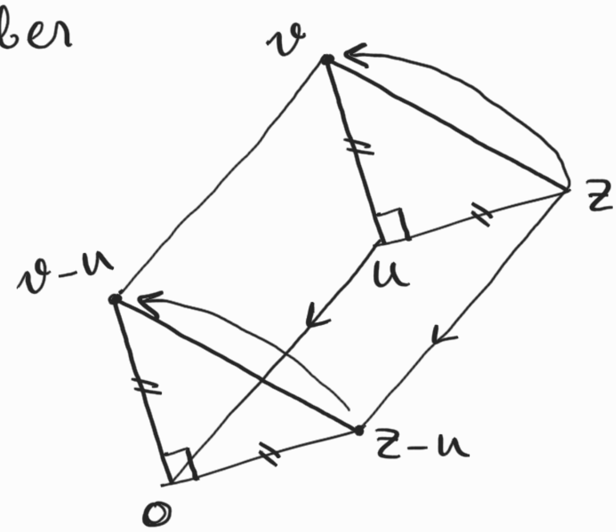


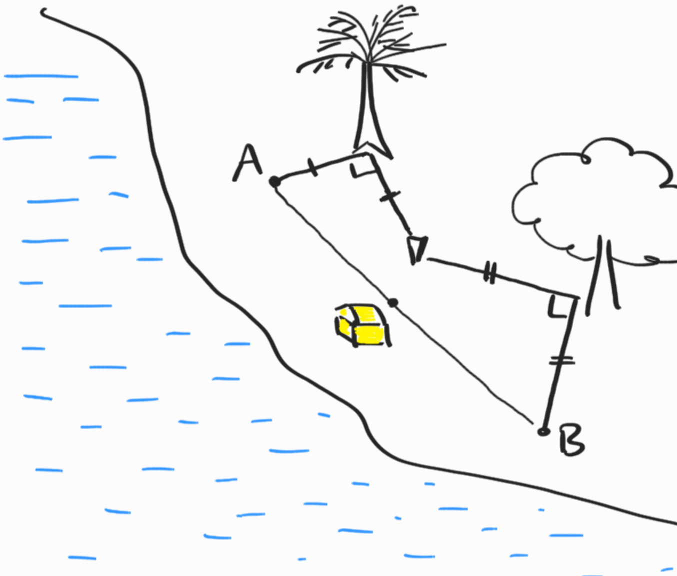


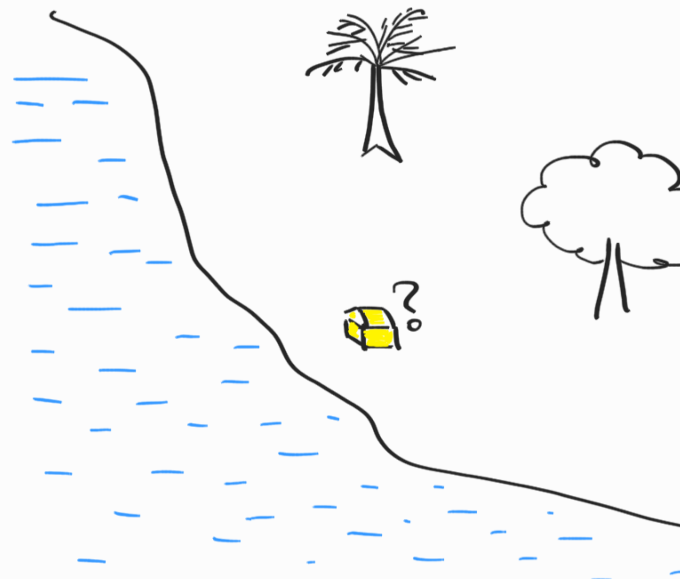
Rotating a complex number 90° anticlockwise about another complex number:

$$v - u = (z - u) \times i$$

$$\therefore \; v = u + (z-u) \times i$$

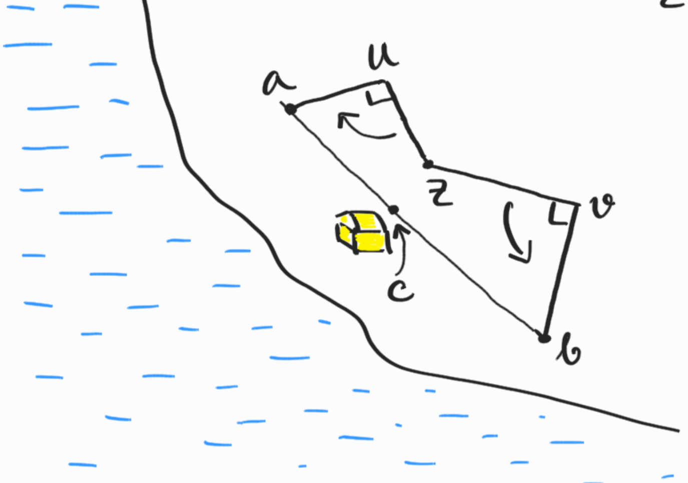






$$(a-u) = (z-u)(-i)$$

$$(b-v) = (z-v)\,i$$



$$a = u - i(z-u)$$

$$b = v + i(z-v)$$

$$c = \frac{a+b}{2} = \frac{u+v}{2} + i\,\frac{u-v}{2}.$$

$\therefore$ c does not depend on z.