Nov 29, 2020

Complex Numbers

Recap $z = a + ib$, $a, b \in \mathbb{R}$.

$\bar{z} = a - ib$. $\quad a = \mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}$,

$|z|^2 = a^2 + b^2 \quad b = \mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i}$.

$z\bar{z} = |z|^2$. $\quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ etc.

$|z_1 z_2| = |z_1| \cdot |z_2| \quad |z_1 + z_2| \le |z_1| + |z_2|$ (triangle inequality)

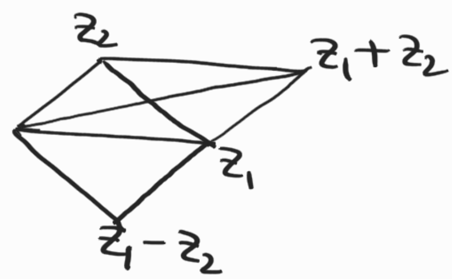
If $z = r(\cos\theta + i\sin\theta)$, then $z_1 z$ is obtained by rotating z_1 anti-clockwise by an angle θ and stretching its modulus by a factor of $|z| = r$.

① Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$$

$$= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2$$

$$= 2(|z_1|^2 + |z_2|^2)$$



② Suppose that $z_1, \dots, z_n$, and $w_1, w_2, \dots, w_n$ are complex numbers inside $\{z \in \mathbb{C} : |z| \le 1\}$.

Show that, $|z_1 z_2 \cdots z_n - w_1 w_2 \cdots w_n|$

$$\le |z_1 - w_1| + |z_2 - w_2| + \dots + |z_n - w_n|.$$

Let us try to show the above for $n=2$:

$$|z_1 z_2 - w_1 w_2|$$

$$= |z_1 z_2 - w_1 z_2 + w_1 z_2 - w_1 w_2|$$

$$= |(z_1 - w_1) z_2 + w_1 (z_2 - w_2)|$$

$$\leq |z_1 - w_1| \cdot |z_2| + |w_1| \cdot |z_2 - w_2| \quad \text{(by triangle ineq.)}$$

$$\leq |z_1 - w_1| + |z_2 - w_2|.$$

Let us try to apply induction on n. We have shown it for $n=2$ (for $n=1$ it also holds, trivially). Now let us do the induction step. Suppose that the assertion holds for all $n \leq k$ $(k \geq 2)$. Let's prove it for $n=k+1$.

$$|\underbrace{z_1 z_2 \cdots z_k}_{z} z_{k+1} - \underbrace{w_1 w_2 \cdots w_k}_{w} w_{k+1}|$$

Note that $|z| \leq 1$, $|w| \leq 1$.

$$\leq |z_1 z_2 \cdots z_k - w_1 w_2 \cdots w_k| + |z_{k+1} - w_{k+1}|$$

(using the case $n=2$)

$$\leq \sum_{j=1}^{k} |z_j - w_j| + |z_{k+1} - w_{k+1}|. \quad \text{(using the case } n=k\text{)}$$

This completes the proof.

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③ Find all $x, y > 0$ satisfying the following:

$$\sqrt{x}\left(1 + \frac{1}{x+y}\right) = \frac{3}{2}, \quad \sqrt{y}\left(1 - \frac{1}{x+y}\right) = \frac{1}{2}.$$

Let $u = \sqrt{x}$, and $v = \sqrt{y}$.

Then the given equations translate to

$$u + \frac{u}{u^2+v^2} = \frac{3}{2}, \quad v - \frac{v}{u^2+v^2} = \frac{1}{2}.$$

(First eqn. + i × second eqn.) gives

$$(u+iv) + \frac{u-iv}{u^2+v^2} = \frac{3}{2} + i\frac{1}{2}$$

$\boxed{\text{Let } z = u+iv}$ $\quad \bar{z}/|z|^2 = 1/z$

$$\iff z + \frac{1}{z} = \frac{3}{2} + i\frac{1}{2}$$

$$\iff 2z^2 - (3+i)z + 2 = 0.$$

$$\iff z = \frac{3+i \pm \sqrt{(3+i)^2 - 4\times 2\times 2}}{2\times 2}$$

$-8+6i$
$= 9i^2 + 6i + 1$
$= (3i+1)^2$

$$= \frac{3+i \pm (3i+1)}{4}$$

$$= 1+i, \ \frac{1-i}{2}.$$

Since $u = \sqrt{x}$, $v = \sqrt{y}$ $(x, y > 0)$, and $z = u+iv$, we conclude that $z = 1+i$, i.e., $(x, y) = (1, 1)$.

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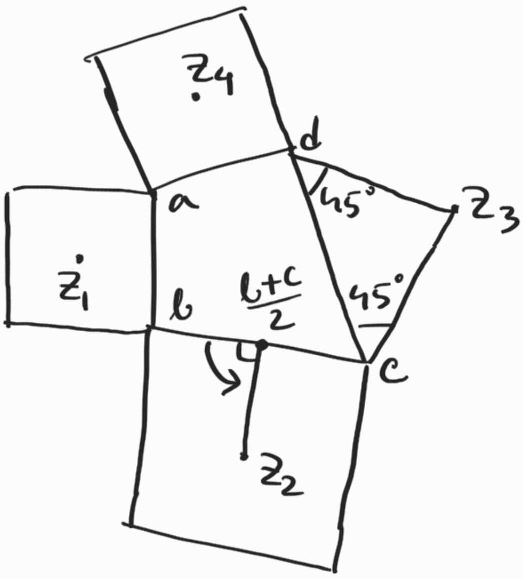
4 On the sides AB, BC, CD, DA of a quadrilateral ABCD, we construct squares exterior to the quad. with centres O_1, O_2, O_3, O_4 respectively. Show that

$$O_1O_3 \perp O_2O_4, \quad O_1O_3 = O_2O_4.$$

Consider the complex coordinates of the points (w.r.t. some coordinate axes and some origin).

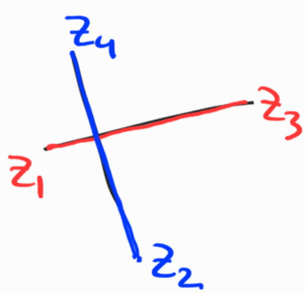
Let a, b, c, d be the complex coordinates of A, B, C, D respectively, and let z_i be the complex coordinate of O_i, for $i = 1, 2, 3, 4$.

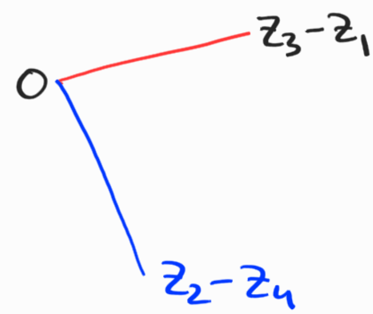


$$z_2 - \frac{b+c}{2} = i\left(b - \frac{b+c}{2}\right)$$

$$\Rightarrow z_2 = \frac{b+c}{2} + i\,\frac{b-c}{2}.$$

Similarly, $z_1 = \frac{a+b}{2} + i\,\frac{a-b}{2}$, etc.





Note that,

$$z_3 - z_1 = \frac{c+d}{2} + i\,\frac{c-d}{2} - \frac{a+b}{2} - i\,\frac{a-b}{2}$$

$$= \frac{c+d-(a+b)}{2} + i\,\frac{(c-d)-(a-b)}{2}.$$

$$z_2 - z_4 = \frac{b+c}{2} + i\,\frac{b-c}{2} - \frac{d+a}{2} - i\,\frac{d-a}{2}$$

$$= \frac{c-d-(a-b)}{2} - i\,\frac{c+d-(a+b)}{2}.$$

Since $z_3 - z_1 = i(z_2 - z_4)$, we are through.

⑤ Let $z_1, z_2, z_3 \in \mathbb{C}$ such that

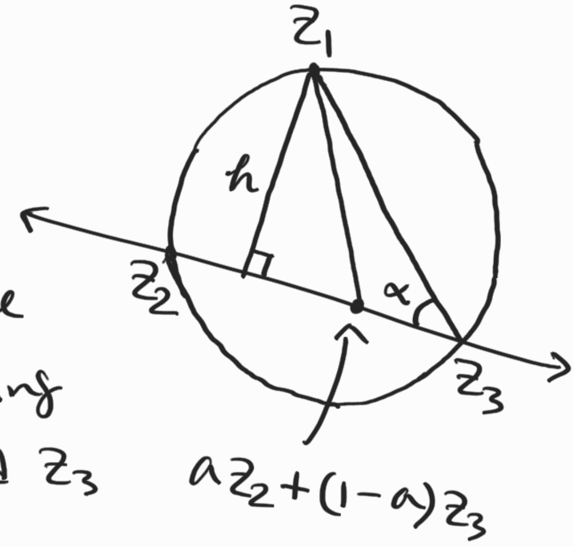
$$|z_1| = |z_2| = |z_3| = R.$$

If $z_2 \neq z_3$, show that

$$\min_{a \in \mathbb{R}} |a z_2 + (1-a) z_3 - z_1| = \frac{1}{2R} |z_1 - z_2| \cdot |z_1 - z_3|.$$

Note that $|z_i| = R$ for each i tells us that z_1, z_2, z_3 are the vertices of a triangle (say, $\begin{matrix} z_1, & z_2, & z_3 \\ \downarrow & \downarrow & \downarrow \\ A & B & C \end{matrix}$), whose circumcircle is the circle with radius R, centred at the origin. And the condition $z_2 \neq z_3$ implies that $R > 0$.

$a z_2 + (1-a) z_3 \longrightarrow$ complex number on the line joining $B(z_2)$ and $C(z_3)$, that divides BC in the ratio $(1-a) : a$. Hence,



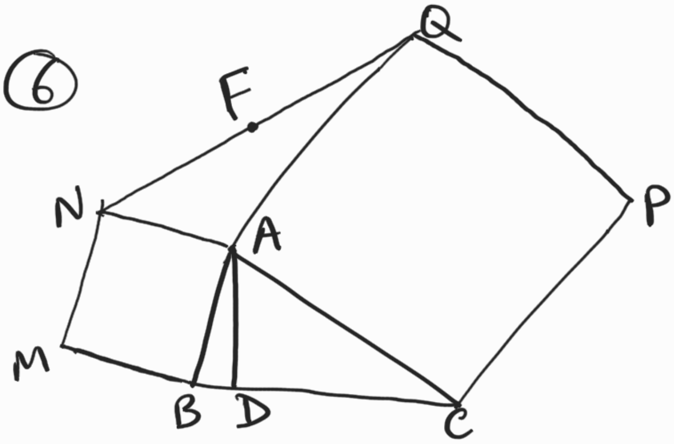
$$\min_{a \in \mathbb{R}} |a z_2 + (1-a) z_3 - z_1|$$

= The perpendicular distance of z_1 from the line joining z_2 and z_3

= h (see the diagram)

$= |z_3 - z_1| \sin\alpha = |z_3 - z_1| \cdot \dfrac{|z_2 - z_1|}{2R}$ (from sine rule). $\square$

⑥

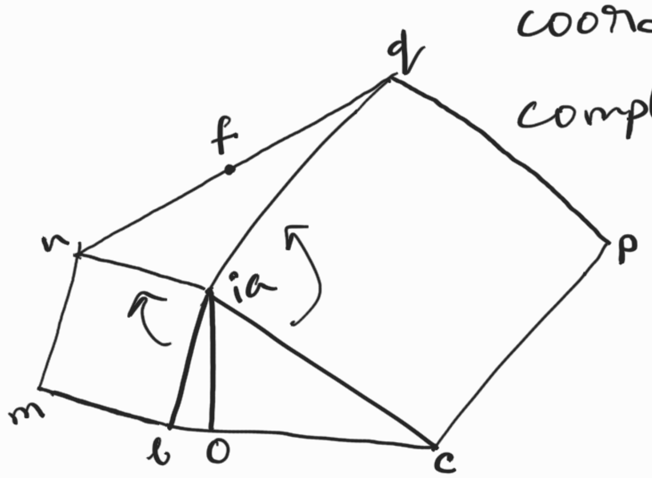


In the adjacent figure, $ABMN$ and $ACPQ$ are squares, and $AD \perp BC$. Suppose that DA, when extended, intersects NQ at E. Show that $NE = EQ$, and $AE = \frac{1}{2} BC$.

Let us set up a complex coordinate system with D as the origin, BC as the real axis, and DA as the imaginary axis.

We proceed indirectly: let F be the midpoint of NQ. We will show that F lies on AD.

For any point X let x denote its complex coordinate, except for A, whose complex coordinate is ia ($a \in \mathbb{R}$).



$$q - ia = i(c - ia)$$

$$\Rightarrow q = a + i(a + c).$$

$$n - ia = -i(b - ia)$$

$$\Rightarrow n = -a + i(a - b).$$

$$\therefore f = \frac{n+q}{2} = i\left(a + \frac{c-b}{2}\right)$$

and $a, b, c \in \mathbb{R} \Rightarrow f$ lies on the imaginary axis (AD).

So, $F = E$. Hence $AE = \frac{f - ia}{i} = \frac{c-b}{2} = \frac{1}{2} BC$. $\square$