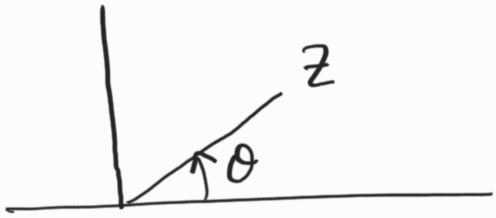
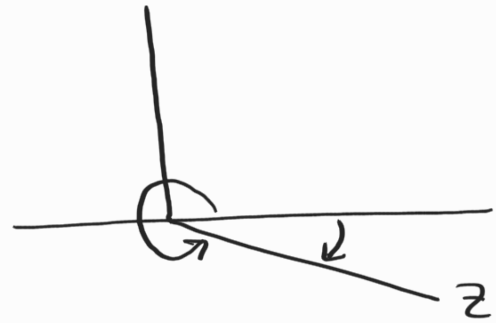
Argument of a complex number

06/12/20





Let $z \in \mathbb{C}$ with $|z| = r$.

$$\arg(z) := \{\theta : z = r(\cos\theta + i\sin\theta)\}$$

e.g., $z = 1 + i\sqrt{3}$. $\arg(z) = \{\frac{\pi}{3} + 2k\pi : k \in \mathbb{Z}\}$.

$\text{Arg}(z)$ = principal argument of z

$= \arg(z) \cap (-\pi, \pi]$.

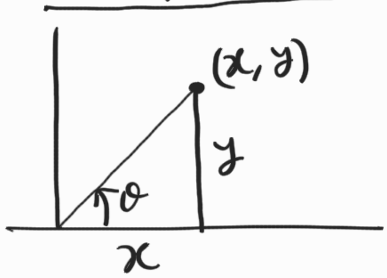
Suppose you are given $z = x + iy$ $(x, y \in \mathbb{R})$.

Then how will you find $\theta = \text{Arg}(z)$?

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \implies \tan\theta = \frac{y}{x} \implies \theta = \tan^{-1}\frac{y}{x}$$

Issue: $\tan^{-1}$ maps $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

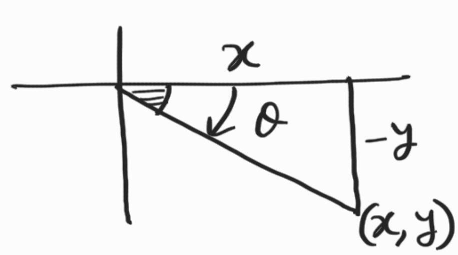
In quad. I



$\theta \in (0, \pi/2)$

$\theta = \tan^{-1}\frac{y}{x}$

In quad. IV



$\tan \measuredangle = \frac{-y}{x}$

$\theta = -\measuredangle$

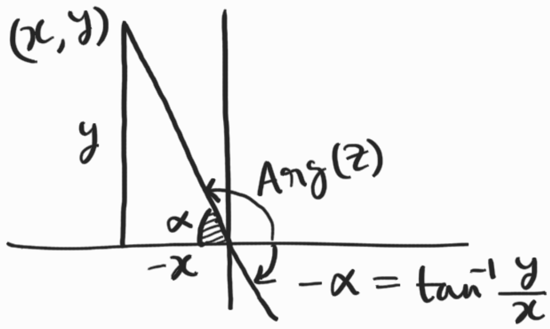
$= -\tan^{-1}\frac{-y}{x}$

$= \tan^{-1}\frac{y}{x}$

$\theta \in (-\pi/2, 0)$

$\theta = \tan^{-1}\frac{y}{x}$

In quad. II

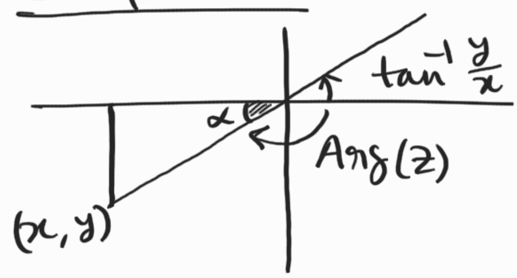


$$\tan\alpha = \frac{y}{-x}$$

$$\Rightarrow \alpha = \tan^{-1}\left(-\frac{y}{x}\right)$$

$$\text{Arg}(z) = \pi - \alpha = \pi + \tan^{-1}\frac{y}{x}$$

In quad. III



$$\tan\alpha = \frac{-y}{-x}$$

$$\text{Arg}(z) = \alpha - \pi$$
$$= -\pi + \tan^{-1}\frac{y}{x}.$$

Let $z = x + iy$ and $\alpha = \tan^{-1}\frac{y}{x}$. Then

$$\text{Arg}(z) = \begin{cases} \alpha & \text{if } x > 0,\ y \in \mathbb{R} \\ \alpha + \pi & \text{if } x < 0,\ y \geq 0 \\ \alpha - \pi & \text{if } x < 0,\ y < 0 \\ \frac{\pi}{2}\,\text{sgn}(y) & \text{if } x = 0,\ y \neq 0 \\ \text{undefined} & \text{if } x = y = 0 \end{cases}$$

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① Express in polar form:

$$1 + \cos a + i\sin a,\quad a \in [0, 2\pi).$$

Sol^n $(1 + \cos a) + i\sin a$

$$= 2\cos\frac{a}{2}\left(\cos\frac{a}{2} + i\sin\frac{a}{2}\right).$$

$$\therefore \text{Modulus}(r) = \left|2\cos\frac{a}{2}\right|.$$

$a \in [0, 2\pi) \Rightarrow a/2 \in [0, \pi)$ $\longrightarrow$ break in two parts, $[0, \pi/2]$ and $(\frac{\pi}{2}, \pi)$.

Case 1: $0 \le a/2 \le \pi/2$.

Here $\cos\frac{a}{2}$, $\sin\frac{a}{2}$ both ≥ 0, so it lies in the first quadrant. Hence, the required principal argument $= \frac{a}{2}$, modulus $= 2\cos\frac{a}{2}$.

Case 2: $\pi/2 < a/2 < \pi$.

$(1+\cos a) + i\sin a$ belongs to the 4th quad.

$= 2\cos\frac{a}{2}\left(\cos\frac{a}{2} + i\sin\frac{a}{2}\right)$

$$\frac{\pi}{2} < \frac{a}{2} < \pi$$

$$-\frac{\pi}{2} < \frac{a}{2} - \pi < 0$$

$$\begin{aligned}\text{Principal argument} &= \tan^{-1}\tan\frac{a}{2} \\ &= \tan^{-1}\tan\left(\frac{a}{2} - \pi\right) \\ &= \frac{a}{2} - \pi.\end{aligned}$$

And modulus: $-2\cos\frac{a}{2}$.

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② Find all $z \in \mathbb{C}$ such that $|z| = 1$, and

$$\left|z/\bar{z} + \bar{z}/z\right| = 1.$$

Given

$z\bar{z} = |z|^2 = 1$ Let $z = \cos\theta + i\sin\theta$.

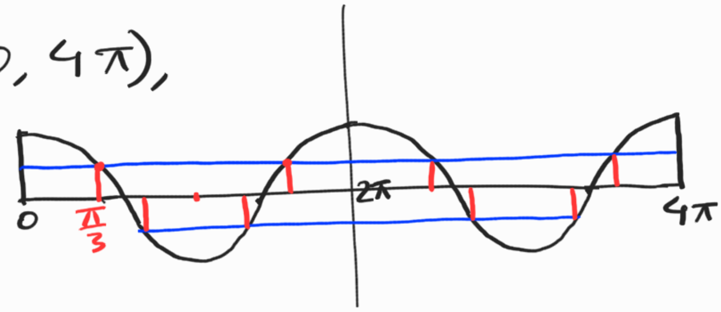
Then,
$$\begin{aligned} z/\bar{z} + \bar{z}/z &= z^2 + \bar{z}^2 \\ &= (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2 \\ &= 2\cos 2\theta. \end{aligned}$$

Thus, $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ becomes equiv. to

$$|2\cos 2\theta| = 1 \quad \text{———} \circledast$$

For $\theta \in [0, 2\pi) \Rightarrow 2\theta \in [0, 4\pi)$,

equation $(\ast)$ has 8 sol^{n}s.



$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= \cos(\theta+\phi) + i\sin(\theta+\phi).$$

For $n \in \mathbb{N}$, and any $\theta \in \mathbb{R}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Note that for $n \in \mathbb{N}$,

$$(\cos\theta + i\sin\theta)^{-n} = \frac{1}{(\cos\theta + i\sin\theta)^n}$$
$$= \frac{1}{\cos n\theta + i\sin n\theta}$$
$$= \cos n\theta - i\sin n\theta$$
$$= \cos(-n\theta) + i\sin(-n\theta).$$

For $n \in \mathbb{Z}$, and any $\theta \in \mathbb{R}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This is known as de Moivre's theorem.

What about $n \in \mathbb{Q}$? Say $n = p/q$.

Let $z = (\cos \tfrac{1}{q}\theta + i \sin \tfrac{1}{q}\theta)$.

$$\left(\cos \tfrac{p}{q}\theta + i \sin \tfrac{p}{q}\theta\right)^q = (z^p)^q = (z^q)^p \quad (p, q \in \mathbb{Z})$$

$$= (\cos\theta + i \sin\theta)^p$$

$$\Rightarrow \cos \tfrac{p}{q}\theta + i \sin \tfrac{p}{q}\theta = (\cos\theta + i\sin\theta)^{p/q}.$$

Issue: $$\underbrace{\cos \tfrac{2\pi}{3} + i \sin \tfrac{2\pi}{3}}_{\neq 1} = (\cos 2\pi + i \sin 2\pi)^{1/3}$$

$$= (\cos 0 + i \sin 0)^{1/3}$$

$$= \cos \tfrac{1}{3}0 + i \sin \tfrac{1}{3}0$$

$$= 1.$$

$$z^3 = 1 \Leftrightarrow (z-1)(z^2+z+1) = 0$$

$$\Leftrightarrow z = 1, \frac{-1 \pm i\sqrt{3}}{2}.$$

$$1^{1/3} = \left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}.$$

$$\underbrace{\cos \tfrac{2\pi}{3} + i \sin \tfrac{2\pi}{3}}_{\neq 1} \in (\cos 2\pi + i \sin 2\pi)^{1/3}$$

$$= \underbrace{(\cos 0 + i \sin 0)^{1/3}}$$

$1 = \cos \tfrac{1}{3}0 + i \sin \tfrac{1}{3}0 \in$ this set.

$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = (z^p)^q = (z^q)^p \quad (p, q \in \mathbb{Z})$$
$$= (\cos\theta + i\sin\theta)^p$$

$$\Rightarrow \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta \in (\cos\theta + i\sin\theta)^{p/q}.$$

In other words, for $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $\gcd(p,q) = 1$,
$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the q many values of $(\cos\theta + i\sin\theta)^{p/q}$.
$q \in \mathbb{N}$

Complex n-th roots of unity

$$z^3 = 1 \Leftrightarrow z = 1, \frac{-1 \pm i\sqrt{3}}{2}.$$

If $\omega = \frac{-1+i\sqrt{3}}{2}$ then $\frac{-1-i\sqrt{3}}{2} = \omega^2$.

If $\omega = \frac{-1-i\sqrt{3}}{2}$ then $\frac{-1+i\sqrt{3}}{2} = \omega^2$.

$$\omega^3 = 1 \Rightarrow (\omega^2)^2 = \omega. \qquad 1 + \omega + \omega^2 = \frac{\omega^3 - 1}{\omega - 1} = 0$$

$$z^n = 1 \Rightarrow |z|^n = 1 \Rightarrow \underbrace{|z| = 1}\ (\because |z| \geq 0)$$
$$\Updownarrow \qquad \text{So, let } z = \cos\theta + i\sin\theta$$

$$\cos n\theta + i\sin n\theta = 1$$

$$\Leftrightarrow \left.\begin{aligned}\cos n\theta &= 1\\ \sin n\theta &= 0\end{aligned}\right\} \Leftrightarrow n\theta = 2k\pi,\ k \in \mathbb{Z}.$$
$$\Leftrightarrow \theta \in \left\{\frac{2k\pi}{n},\ k \in \mathbb{Z}\right\}.$$

$z^n = 1 \iff z = \cos\theta + i\sin\theta$ where

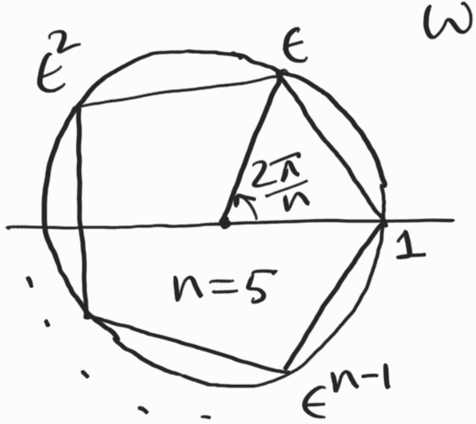
$\theta \in \left\{ \frac{2k\pi}{n},\ k \in \mathbb{Z} \right\}$.

$\iff z \in \left\{ \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\right.$

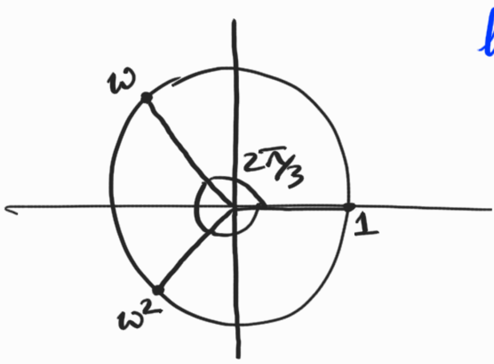
$\left. \text{where } k = 0, 1, \dots, n-1 \right\}$

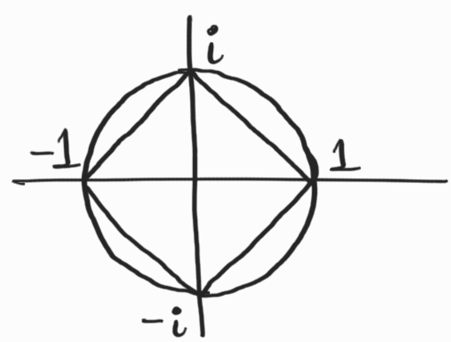
$\iff z \in \{1, \epsilon, \epsilon^2, \dots, \epsilon^{n-1}\}$

where $\epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.



Thus, the complex n^{th} roots of unity are the vertices of a regular n-gon, inscribed in the circle $|z| = 1$, and having 1 as one of its vertices.





<u>Problem</u> If $z_1, \dots, z_n$ are the complex n^{th} roots of unity, find $z_1^k + z_2^k + \dots + z_n^k$, where $k \in \mathbb{N}$.

We know, if $\epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then we can write

$$\{z_1, \dots, z_n\} = \{1, \epsilon, \epsilon^2, \dots, \epsilon^{n-1}\}.$$

Therefore,

$$\sum_{j=1}^{n} z_j^k = 1 + \epsilon^k + \epsilon^{2k} + \dots + \epsilon^{(n-1)k}$$

$$= \begin{cases} \dfrac{1-\epsilon^{nk}}{1-\epsilon^k} & \text{if } \epsilon^k \neq 1 \\ n & \text{if } \epsilon^k = 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \epsilon^k \neq 1 \\ n & \text{if } \epsilon^k = 1 \end{cases} \quad \dots (*)$$

Here $\epsilon^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, which equals 1 if and only if

$$\frac{2k\pi}{n} = 2l\pi \text{ for some } l \in \mathbb{Z}$$

$$\iff k = ln \text{ for some } l \in \mathbb{Z}$$

$$\iff n \mid k.$$

Thus, $\epsilon^k = 1 \iff n \mid k$ here. [In general, $\epsilon^n = 1$, $\epsilon^k = 1 \not\Rightarrow n \mid k$.]

Hence, $(*)$ reduces to

$$\sum_{j=1}^{n} z_j^k = \sum_{j=0}^{n-1} \epsilon^{jk} = \begin{cases} 0 & \text{if } n \nmid k, \\ n & \text{if } n \mid k. \end{cases} \text{ (Ans)}$$