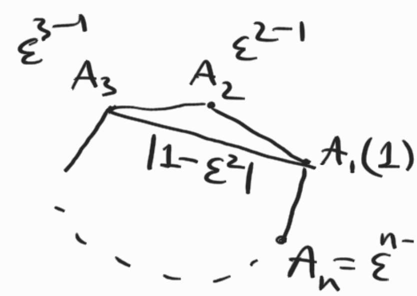
13/12/2020

① Suppose that $A_1 A_2 \cdots A_n$ is a regular n-gon, inscribed in a circle of radius 1.

Determine the product

$$\overline{A_1A_2} \times \overline{A_1A_3} \times \cdots \times \overline{A_1A_n}.$$

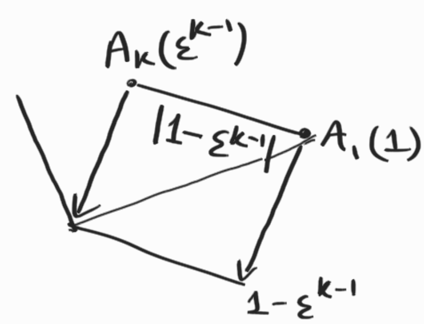
Sol^n We can consider a complex coordinate system in which $A_1 = 1$ and $A_1, A_2, \cdots, A_n$ have complex coordinates same as the n complex n-th roots of unity. In other words, A_k will have complex coordinate ε^{k-1}, where $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Hence,



$$\prod_{k=2}^{n} \overline{A_1A_k} = \prod_{k=2}^{n} |1-\varepsilon^{k-1}|$$

$$= |1-\varepsilon|\cdot|1-\varepsilon^2|\cdots|1-\varepsilon^{n-1}|$$

$$= |(1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n-1})|$$



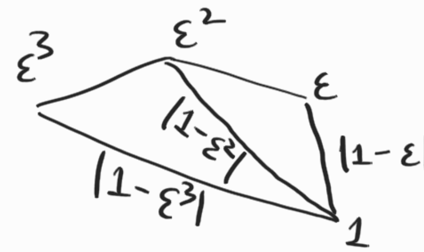
Since $1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{n-1}$ are the roots of the poly. $x^n - 1$, we can say that

$$x^n - 1 \equiv (x-1)(x-\varepsilon)(x-\varepsilon^2)\cdots(x-\varepsilon^{n-1})$$

$$\Rightarrow (x-\varepsilon)(x-\varepsilon^2)\cdots(x-\varepsilon^{n-1}) \equiv \frac{x^n-1}{x-1}$$

for all $x \neq 1$.

$(1 + x + x^2 + \cdots + x^{n-1})$



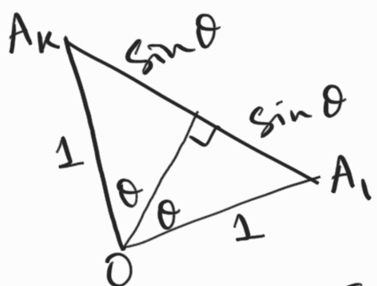
$$(x-\varepsilon)(x-\varepsilon^2)\cdots(x-\varepsilon^{n-1}) = 1 + x + x^2 + \cdots + x^{n-1} \quad (*)$$

Since two Poly.s equal at infinitely points implies that they must be identical, we can say that $(*)$ holds for every $x \in \mathbb{C}$. Hence, we can put $x = 1$ in $(*)$ to get

$$(1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n-1}) = n.$$

Therefore, the desired product should be n.

<u>Remark.</u> $\quad \overline{A_1A_K} = 2\sin\left(\frac{1}{2}\angle A_1 O A_K\right) = 2\sin\frac{\pi(k-1)}{n}.$



So it follows from what we just derived, that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

$$\prod_{k=2}^{n}\sin\frac{\pi(k-1)}{n} = \frac{1}{2^{n-1}}\prod_{k=2}^{n} 2\sin\frac{(k-1)\pi}{n}$$

$$= \frac{1}{2^{n-1}}\prod_{k=2}^{n}\overline{A_1A_K} = \frac{n}{2^{n-1}}.$$

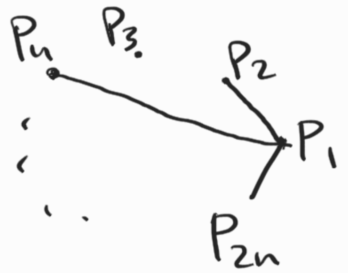
<u>Evaluate</u>: $\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\cdots\sin\frac{(2n-1)\pi}{2n}.$

If $P_1P_2\cdots P_{2n}$ is a regular $2n$-gon, then $P_1P_3\cdots P_{2n-1}$ is a regular n-gon.

$$\prod_{j=1}^{n} \sin\frac{(2j-1)\pi}{2n} = \prod_{k=1}^{2n-1} \sin\frac{k\pi}{2n} \Big/ \prod_{j=1}^{n-1} \sin\frac{\cancel{2}j\pi}{\cancel{2}n}$$

$$= \frac{2n/2^{2n-1}}{n/2^{n-1}} = \frac{1}{2^{n-1}}.$$

Alt. Sol^n If $P_1P_2\cdots P_{2n}$ is a regular $2n$-gon, then $P_1P_3\cdots P_{2n-1}$ is a regular n-gon.



$$\prod_{j=1}^{n} 2\sin\frac{(2j-1)\pi}{2n} = \overline{P_1P_2}\;\overline{P_1P_4}\cdots\overline{P_1P_{2n}}$$

$$= \overline{P_1P_2}\;\overline{P_1P_3}\cdots\overline{P_1P_{2n}} \Big/ \overline{P_1P_3}\;\overline{P_1P_5}\cdots\overline{P_1P_{2n-1}}$$

$$= 2n/n = 2$$

② A sequence $(a_1, b_1), (a_2, b_2), \ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = (\sqrt{3}\,a_n - b_n, \sqrt{3}\,b_n + a_n)$$

for each $n = 1, 2, 3, \ldots$. If $(a_{100}, b_{100}) = (18, 20)$, what is $a_1 + b_1$?

Sol^n Let $z_n = a_n + ib_n$. Note that,

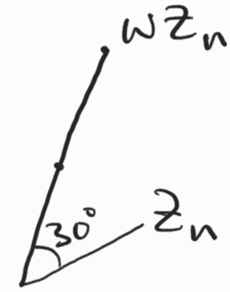
$$z_{n+1} = (\sqrt{3}\,a_n - b_n) + i(\sqrt{3}\,b_n + a_n)$$

$$= (\sqrt{3} + i)\,z_n$$

Thus,

$$z_{n+1} = w\, z_n, \text{ for all } n \geq 1,$$

where $w = \sqrt{3} + i$.



$$\therefore\ z_n = w^{n-1} z_1 = 2^{n-1}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{n-1} z_1$$

$$\Rightarrow z_1 = 2^{-99}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)^{99} z_{100}$$

$$= 2^{-99}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)^{99} z_{100}$$

$$= 2^{-99}\left[\cos\left(-\frac{99\pi}{6}\right) + i\sin\left(-\frac{99\pi}{6}\right)\right] z_{100}$$

$$= 2^{-99}\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right](18 + 20i)$$

$$= 2^{-99}(20 - 18i).$$

$$\therefore\ a_1 + b_1 = 2^{-99}(20 - 18) = 2^{-98}.$$

$\underline{e^{i\theta} = \cos\theta + i\sin\theta}$, $\theta \in \mathbb{R}$. Put $\theta = \frac{\pi}{2}$.

$$i = e^{i\pi/2} \Rightarrow i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}.$$

$$i = e^{i5\pi/2} \Rightarrow i^i = \left(e^{i5\pi/2}\right)^i = e^{-5\pi/2}.$$

$$i^i = \{e^{-\pi/2 + 2k\pi} : k \in \mathbb{Z}\}.$$

How to define a^b for $a, b \in \mathbb{C}$?

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \dots = \cos\theta + i\sin\theta.$$

You will see a full justification in a course on Complex Analysis.

$a, b \in \mathbb{C}$. $a = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

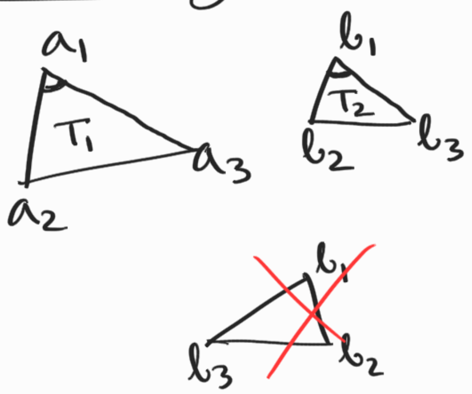
$\log a := \log r + i\theta$ → $\arg(a)$: a set.

$\operatorname{Log} a := \log|a| + i\,\underline{\operatorname{Arg}(a)}$ → Principal arg.

$$a^b = e^{b \log a} = e^{c+id} = \underbrace{e^c(\cos d + i\sin d)}_{\text{a set actually}}$$

$b\log a$ ↓ set → a set actually

<u>Similarity</u>



$T_1 \sim T_2$ (with same <u>orientation</u>) if and only if

(i) $\dfrac{|a_2 - a_1|}{|b_2 - b_1|} = \dfrac{|a_3 - a_1|}{|b_3 - b_1|}$, and

(ii) $\arg\left(\dfrac{a_3 - a_1}{a_2 - a_1}\right) = \arg\left(\dfrac{b_3 - b_1}{b_2 - b_1}\right)$

(i) + (ii) is equiv. to

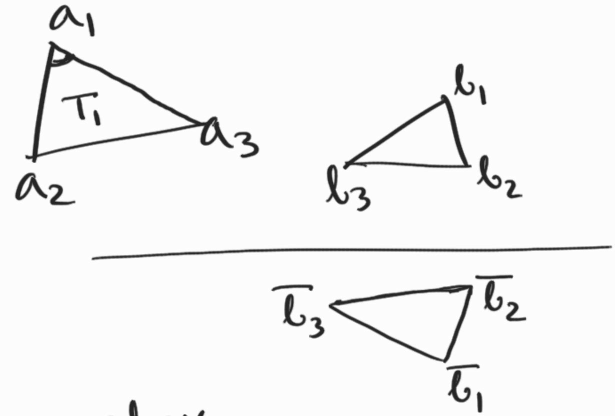
$$\frac{a_3 - a_1}{a_2 - a_1} = \frac{b_3 - b_1}{b_2 - b_1}.$$

$a_1a_2a_3 \sim b_1b_2b_3$ with same orientation iff $\dfrac{a_2 - a_1}{a_3 - a_1} = \dfrac{b_2 - b_1}{b_3 - b_1}$.

$a_1a_2a_3 \sim b_1b_2b_3$

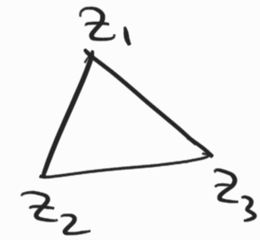
with opposite orientation

iff $\dfrac{a_2-a_1}{a_3-a_1} = \dfrac{\bar{b}_2-\bar{b}_1}{\bar{b}_3-\bar{b}_1}$.



<u>Fact</u> Let z_1, z_2, z_3 be the complex coordinates of the vertices of a triangle T. Then, the following are all equivalent:

(i) T is equilateral,



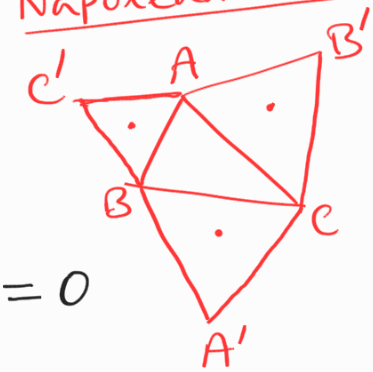
(ii) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

(iii) $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{z_3 - z_2}{z_1 - z_2}$ $\Longleftrightarrow$ $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$

(iv) $z_1^2 + z_2^2 + z_3^2$
$= z_1z_2 + z_2z_3 + z_3z_1$

Show that these are also equivalent to

Napolean's thm



(v) $z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$

(vi) $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$.

(vii) $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$

where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

Correction: (v) <=> (vi) => (iv) <=> (i), but (i) need not imply (v) or (vi). In words, (v) and (vi) are equivalent to each other, but they only imply that the triangle is equilateral, not the other way around.