Complex Numbers: Day 5

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In today's class we will see some more applications of complex numbers.

Problem 1. Suppose that z_1, z_2, z_3 are the vertices of a triangle and $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$. Show that the triangle must be equilateral.

Solution. Since $|z_1| = |z_2| = |z_3|$, the points z_1, z_2, z_3 all lie on a circle centred at the origin. On the other hand, the centroid of the triangle is has complex coordinate $\frac{1}{3}(z_1+z_2+z_3) = 0$. Thus, the origin is both the circumcentre and the centroid of the triangle. We know that if the centroid and the circumcentre of a triangle coincide, then it must be equilateral. □

Problem 2. Determine the value of $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}$.

Solution. Let $z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$. Then

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \mathrm{Re}(z - z^2 + z^3). \tag{1}$$

(This follows from de Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.) Again, by de Moivre's theorem, $z^7 = \cos\pi + i\sin\pi = -1$. Therefore, z is a complex root of the equation $z^7 + 1 = 0$. Since $z \neq -1$, we can factor it out and cancel the factor $(z+1)$.

$$\begin{aligned}
0 = z^7 + 1 &= (z+1)(1 - z + z^2 - z^3 + z^4 - z^5 + z^6) \\
\implies 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 &= 0 \\
\implies 1 - (z - z^2 + z^3) + z^3(z - z^2 + z^3) &= 0 \\
\implies (1 - z^3)(z - z^2 + z^3) &= 1.
\end{aligned}$$

Combining this with (1), we get

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \mathrm{Re}\left(\frac{1}{1-z^3}\right).$$

Let's call $z^3 = w$. Since $|w| = 1$, we have $\overline{w} = 1/w$, and hence

$$\mathrm{Re}\left(\frac{1}{1-w}\right) = \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1-\overline{w}}\right) = \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1-1/w}\right) = \frac{1}{2}.$$

Hence the value of the required expression is $1/2$. □

Problem 3. Calculate the sum $\sum_{k=0}^{n} \binom{n}{k} \cos k\alpha$ where $\alpha \in [0, \pi]$.

Solution. Let $A_n = \sum_{k=0}^{n} \binom{n}{k} \cos k\alpha$, and $B_n = \sum_{k=0}^{n} \binom{n}{k} \sin k\alpha$. Then,

$$\begin{aligned}
A_n + iB_n &= \sum_{k=0}^{n} \binom{n}{k} (\cos k\alpha + i \sin k\alpha) \\
&= \sum_{k=0}^{n} \binom{n}{k} (\cos \alpha + i \sin \alpha)^k && \text{(by de Moivre's theorem)} \\
&= (1 + (\cos \alpha + i \sin \alpha))^n && \text{(by binomial theorem)} \\
&= \left(2 \cos \frac{\alpha}{2}\right)^n \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2}\right)^n \\
&= \left(2 \cos \frac{\alpha}{2}\right)^n \left(\cos \frac{n\alpha}{2} + i \sin \frac{n\alpha}{2}\right). && \text{(by de Moivre's theorem)}
\end{aligned}$$

Therefore,

$$A_n = \left(2 \cos \frac{\alpha}{2}\right)^n \cos \frac{n\alpha}{2}.$$

Note that as a by-product we also got an expression for the B_n defined above. □

Problem 4. Suppose that a, b, c are complex numbers such that all the roots of the equation $x^3 + ax^2 + bx + c = 0$ have equal absolute value (modulus if they are complex). Prove that $a = 0 \iff b = 0$.

Solution. Let z_1, z_2, z_3 be roots. We are given that $|z_1| = |z_2| = |z_3| = r$, say. Note that if $r = 0$ then $z_1 = z_2 = z_3 = 0$, and the conclusion holds. So let us now assume that $r > 0$. From Vieta's theorem, $z_1 + z_2 + z_3 = -a$ and $z_1z_2 + z_2z_3 + z_3z_1 = b$. Hence,

$$\begin{aligned}
a = 0 &\iff z_1 + z_2 + z_3 = 0 \\
&\iff \overline{z_1} + \overline{z_2} + \overline{z_3} = 0 \\
&\iff \frac{r^2}{z_1} + \frac{r^2}{z_2} + \frac{r^2}{z_3} = 0 \\
&\iff z_1z_2 + z_2z_3 + z_3z_1 = 0 \\
&\iff b = 0.
\end{aligned}$$

This completes the proof. □

Problem 5. Let $n \geq 3$ be an integer and a be any non-zero real number. Prove that any non-real root of the equation $x^n + ax + 1 = 0$ must satisfy the inequality

$$|z| \geq \frac{1}{\sqrt[n]{n-1}}.$$

Solution. Let $z = r(\cos\theta + i\sin\theta)$, where $\theta \in (-\pi, \pi]$. Since we are working with only the non-real roots of the equation, we assume $\theta \neq 0, \pi$. Plugging this z into the given equation, and using de Moivre's theorem, we get

$$r^n(\cos n\theta + i\sin n\theta) + ar(\cos\theta + i\sin\theta) + 1 = 0$$

which is equivalent to the following equations:

$$r^n \cos n\theta + ar\cos\theta = -1, \quad \text{and} \quad r^n \sin n\theta + ar\sin\theta = 0.$$

We eliminate r, by subtracting $\cos\theta$ times the second equation from $\sin\theta$ times the first equation, and get

$$r^n\left(\cos n\theta \sin\theta - \sin n\theta\cos\theta\right) = -\sin\theta \implies r^n \sin(n-1)\theta = \sin\theta.$$

We can prove by induction that the inequality $|\sin k\theta| \leq k|\sin\theta|$ holds for all $k \in \mathbb{N}$ and for every $\theta \in (-\pi, \pi]$. (check!) Hence, $|\sin(n-1)\theta| \leq (n-1)|\sin\theta|$, which in turn implies that

$$|\sin\theta| = r^n|\sin(n-1)\theta| \leq (n-1)r^n|\sin\theta|.$$

Since $\sin\theta \neq 0$, we conclude that $r^n \geq 1/(n-1)$, as required. □

Hint for the induction:

$$\begin{aligned}|\sin(k+1)\theta| &= |\sin k\theta\cos\theta + \cos k\theta\sin\theta| \\ &\leq |\sin k\theta\cos\theta| + |\cos k\theta\sin\theta| \\ &\leq |\sin k\theta| + |\sin\theta|.\end{aligned}$$

Problem 6. On the sides AB and AC of $\triangle ABC$ equilateral triangles ABN and ACM are constructed, external to the triangle. If P, Q, R be the midpoints of the segments BC, AM, and AN respectively, show that $\triangle PQR$ is equilateral.

Solution. Looking at Figure 1, let us consider a complex coordinate system with point A being the origin. Let us denote by x the complex coordinate of a point X. Consider the complex number $\varepsilon = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. Since multiplication by ε results in a $60°$ anticlockwise rotation about the origin,

$$m = c\varepsilon, \text{ and } n = b/\varepsilon.$$

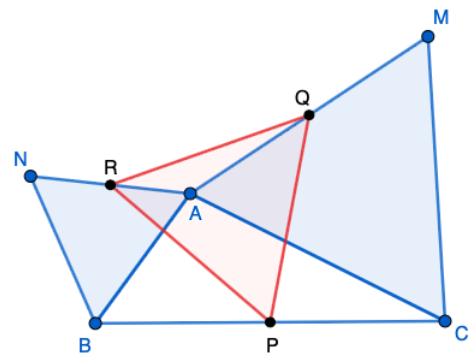

Figure 1: Diagram for Problem 6

Hence the coordinates of the vertices of $\triangle PQR$ are given by

$$p = \frac{b+c}{2}, \quad q = \frac{c}{2}\varepsilon, \quad r = \frac{b}{2}\frac{1}{\varepsilon}.$$

We know that if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$then z_1, z_2, z_3 form the vertices of an equilateral triangle. Hence it suffices to show here that

$$p^2 + q^2 + r^2 = pq + qr + rp. \tag{2}$$

To show this, first note that $\varepsilon^3 + 1 = 0, \varepsilon \neq -1 \implies \varepsilon^2 - \varepsilon + 1 = 0$. We shall use the relation $1 + \varepsilon^2 = \varepsilon$ to simplify both the LHS and RHS of (2), as follows.

$$\begin{aligned}
pq + qr + rp &= \frac{bc}{4}\varepsilon + \frac{c^2}{4}\varepsilon + \frac{bc}{4} + \frac{b^2}{4}\frac{1}{\varepsilon} + \frac{bc}{4}\frac{1}{\varepsilon} \\
&= \frac{bc}{4} \cdot \frac{\varepsilon^2 + \varepsilon + 1}{\varepsilon} + \frac{c^2}{4}\varepsilon + \frac{b^2}{4}\frac{1}{\varepsilon} \\
&= \frac{bc}{4} \cdot \frac{2\not\varepsilon}{\not\varepsilon} + \frac{c^2}{4}\varepsilon + \frac{b^2}{4}\frac{1}{\varepsilon}. \qquad (\text{since } \varepsilon^2 + 1 = \varepsilon)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
p^2 + q^2 + r^2 &= \frac{(b+c)^2}{4} + \frac{c^2}{4}\varepsilon^2 + \frac{b^2}{4}\frac{1}{\varepsilon^2} \\
&= \frac{bc}{2} + \frac{c^2}{4}(\varepsilon^2 + 1) + \frac{b^2}{4} \cdot \frac{1 + \varepsilon^2}{\varepsilon^2} \\
&= \frac{bc}{2} + \frac{c^2}{4}\varepsilon + \frac{b^2}{4}\frac{\not\varepsilon}{\varepsilon^{\not 2}}. \qquad (\text{since } \varepsilon^2 + 1 = \varepsilon)
\end{aligned}$$

Therefore the LHS and RHS of equation (2) are equal, which completes the proof. □
