08/05/21

Complex Numbers

Historical Intro:

Tartaglia: Can you find a number (say x) such that

$$x^3 + px = q \, ?$$

(p, q given numbers)

(Secret formula:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \, . \Big)$$

Cardano: Find x s.t. $x^3 = px + q$?

Another Qs: Can you divide 10 into two parts whose product is 40?

$$\boxed{\begin{array}{c} x + y = 10 \\ xy = 40 \end{array}}$$

People: No, not possible.

Cardano: Wrong, take $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$.

$$(5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10$$

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 5^2 - (-15) = 40.$$

Bombelli: $\boxed{x^3 = 15x + 4.}$ $\qquad x^3 = px + q$

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} \, .$$

Cardano

As per this formula, we get

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \, .$$

Common man's sol^n: $x = 4$.

Bombelli: $\sqrt[3]{2+\sqrt{-121}} = 2+\sqrt{-1}$

and $\sqrt[3]{2-\sqrt{-121}} = 2-\sqrt{-1}$.

$(2+\sqrt{-1})^3$
$= \cdots$ Simplify using usual algebra.

So, the sol^{n}s are exactly the same.

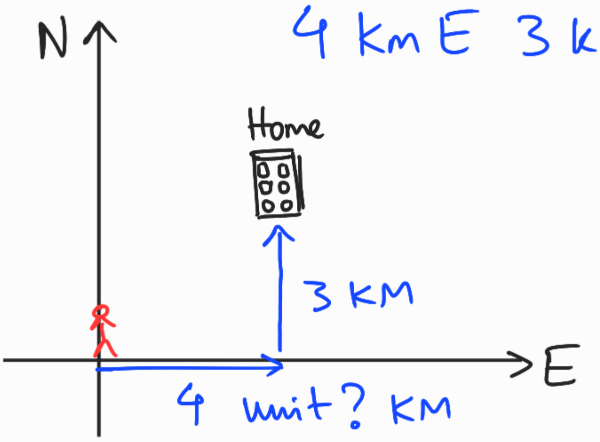
Textbook Intro:

"Call $i=\sqrt{-1}$", define $\mathbb{C} = \{a+ib : a, b \in \mathbb{R}\}$.

"Assume" that all the rules of algebra, such as $u+v = v+u$, $(u+v)c = uc+vc$, etc. also work with this new set of numbers.

Shortcoming of this approach: we do not get to visualize, especially multiplication of complex no.s.

Geometrical Intro:



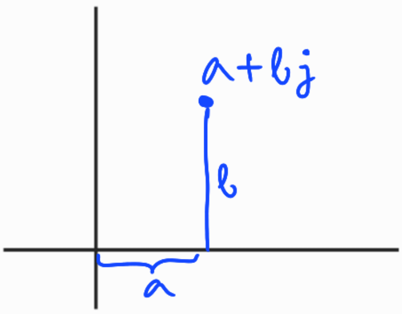
4 Km E 3 km N: $4\times(1\text{ km E})+3\times(1\text{ km N})$

$4\hat{i}+3\hat{j}$ (unit: KM) ← vector notation

$(4, 3)$ ← coordinates

$= 4\,(1,0)+3\,(0,1)$

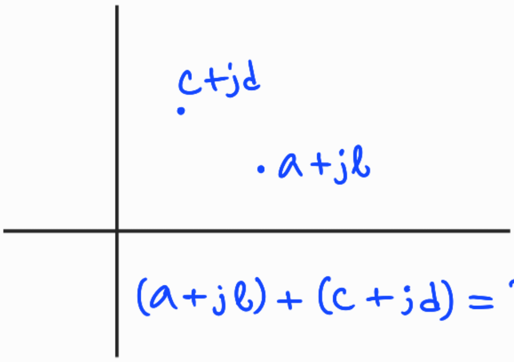
$4\times1+3j$ ← complex no.



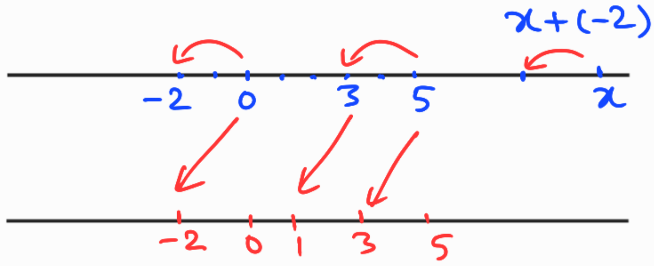
(We'll see later that $j^2=-1$)

j : unit for "y" axis

1 : unit for "x" axis



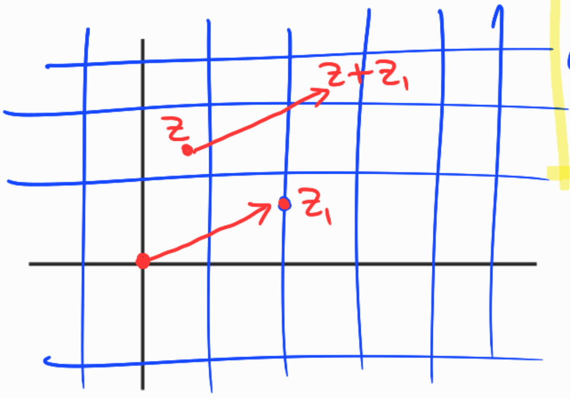
$(a+jb)+(c+jd)=?$

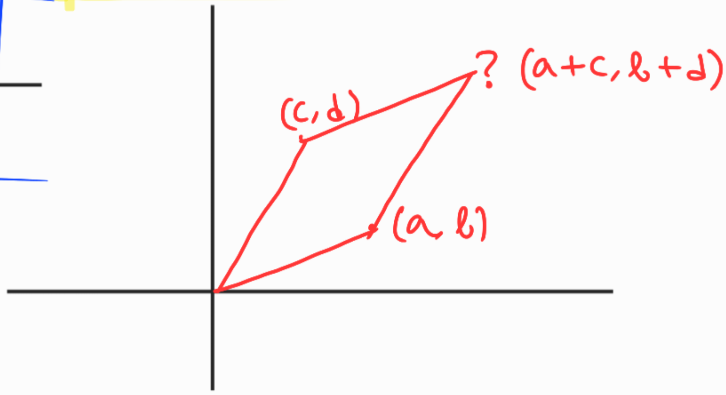


add (−2) : shift the real line s.t. 0 goes to (−2).

add z_1 : shift the axes so that 0 goes to z_1

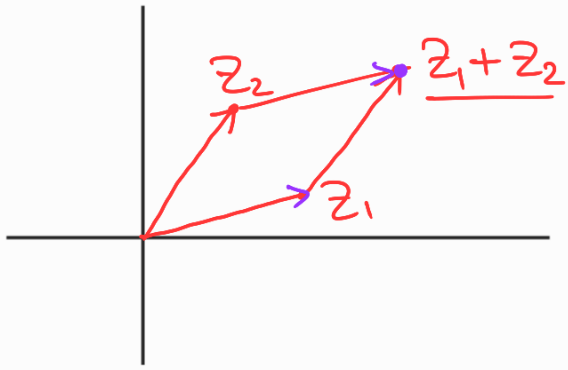
$z_1 = a + jb$



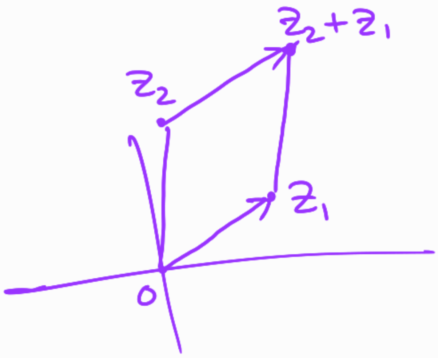


$(c+jd)+(a+jb)$

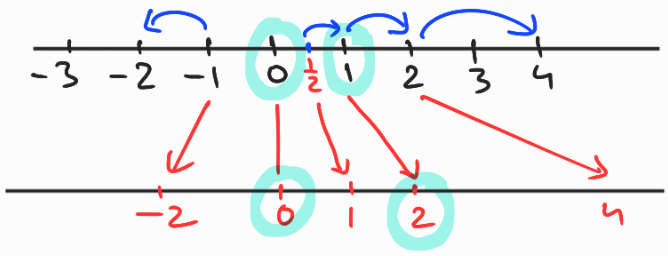
$= (c+a) + j(d+b)$



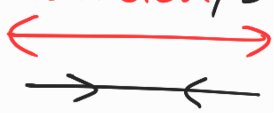
$z_1 + z_2 = z_2 + z_1$



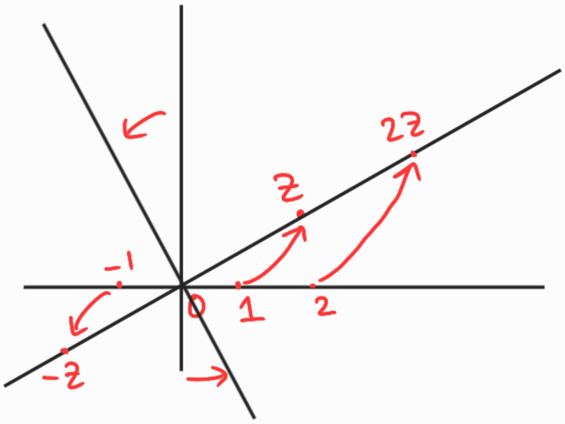
×2 → interpretation?



Stretch/Squeeze

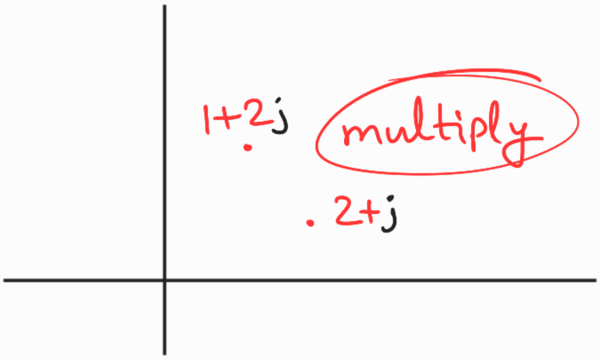


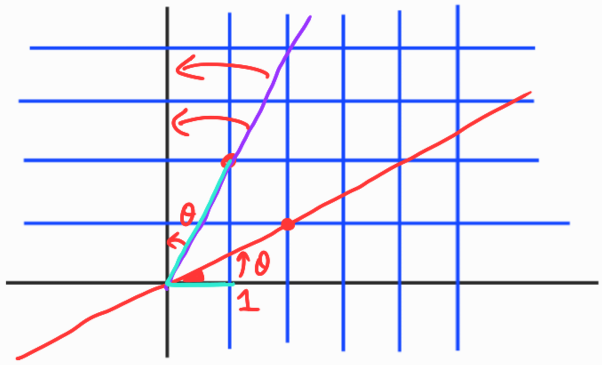
multiply by x: stretch/squeeze real line about the origin s.t. 1 goes to x.



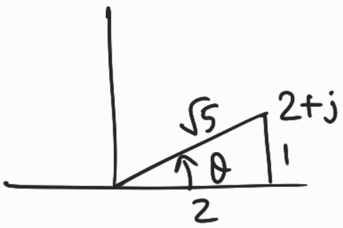
multiplication by z

rotate + stretch/shrink the axes about the origin s.t. 1 goes to z.



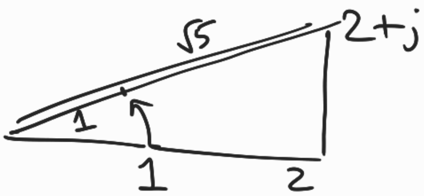


mult. by (2+j) : rotate by angle + stretch s.t. 1 goes to 2+j

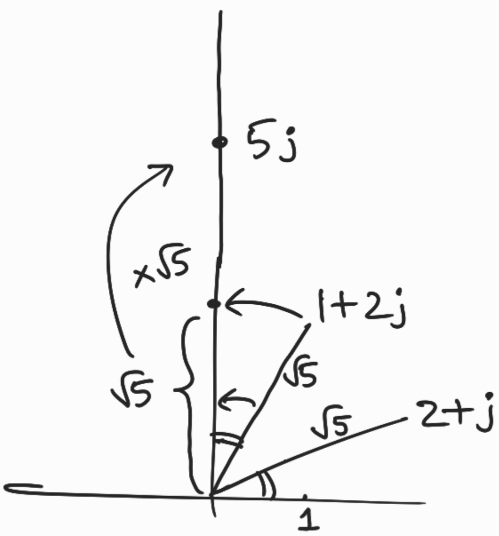


rotate anticlockwise by an angle θ where θ is shown

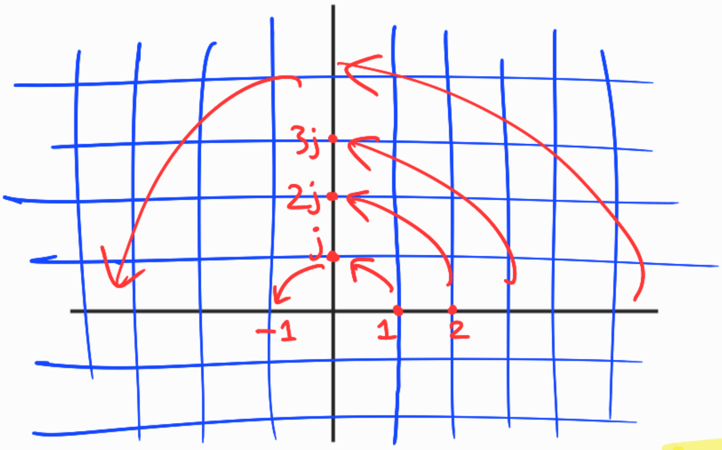
and stretch by factor √5.



$$(1+2j)(2+j)$$
$$= 5j$$



multiplication by j?



$1 \times j = j$

$2 \times j = 2j$

$x \times j =$ rotate x 90° around origin (anticlockwise)

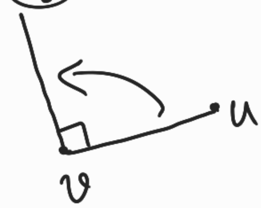
$\boxed{j \times j = -1}$

This is why we say that $\sqrt{-1}$ is the unit for the y-axis in the plane of complex numbers.

So, from now on, we'll write $a + ib$ and keep in mind that $i = \sqrt{-1}$ (or, that $i^2 = -1$)

multiply z by i $\rightarrow$ rotate z by 90° (anticlockwise) around the origin

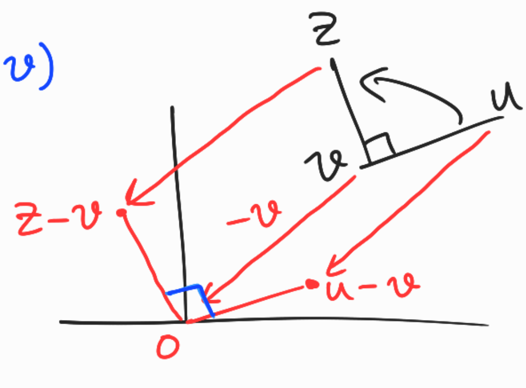
②



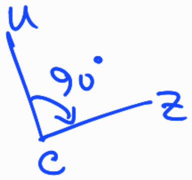
$z - v = i(u - v)$

$\downarrow$

$z = v + i(u - v).$



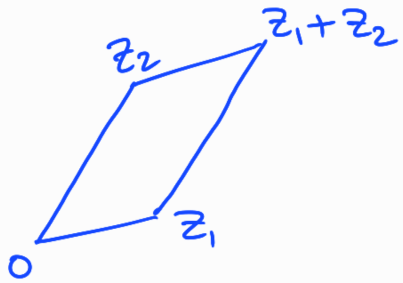
Clockwise?



$$u - c = i(z - c)$$

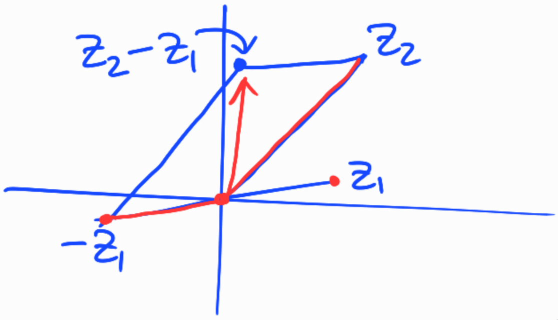
$$\Rightarrow -i(u-c) = z - c$$

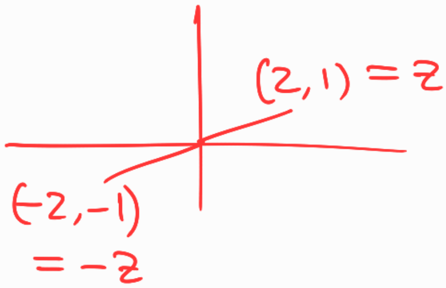
$$\frac{1}{i} = -\frac{i^2}{i} = -i \qquad \Rightarrow \underline{z = c - i(u-c)}$$

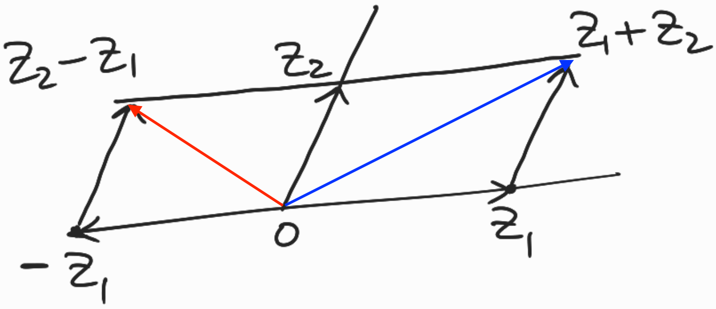


$z_2 - z_1 \rightarrow$ where is it?

$= z_2 + (-z_1)$







A fun problem

(How to find the treasure?)