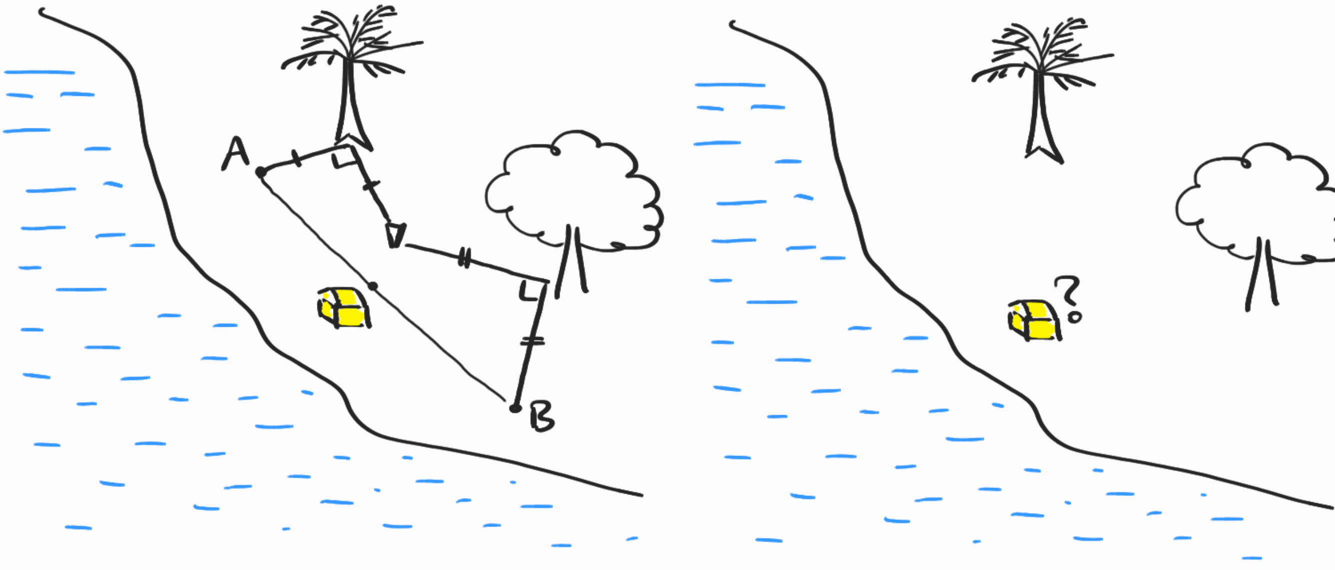
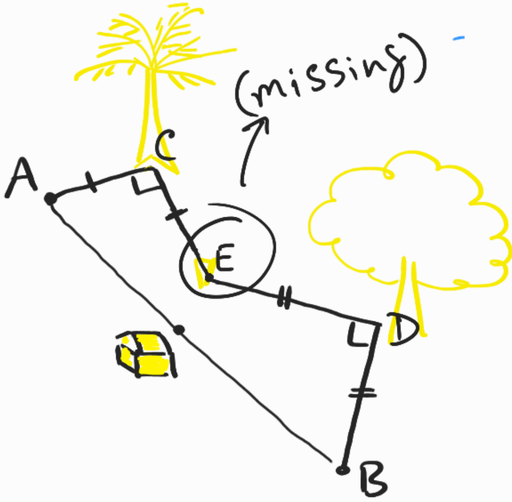
A fun problem

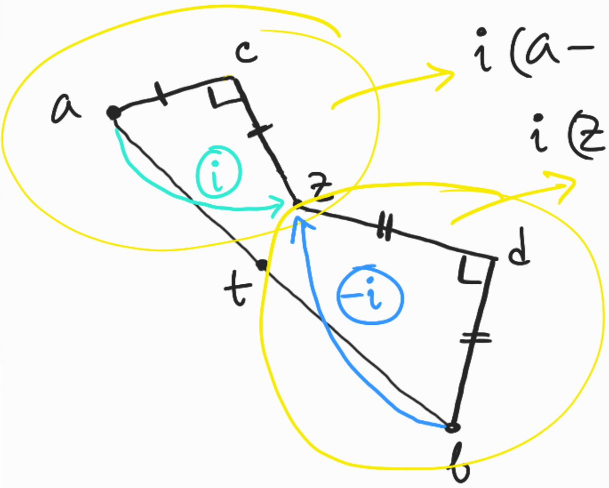
(How to find the treasure?)



Let us set up a complex coord. system with any choice of the origin and the axes.

For a generic point X let x denote its complex coordinate.





$$i(a-c) = z-c$$

$$i(z-d) = b-d$$

$$\boxed{t = \frac{a+b}{2}}$$

$$b = d + i(z-d)$$

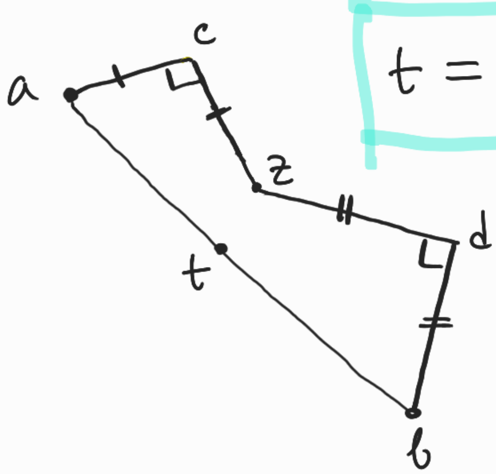
$$a = c - i(z-c)$$

$$\text{So, } t = \frac{a+b}{2}$$

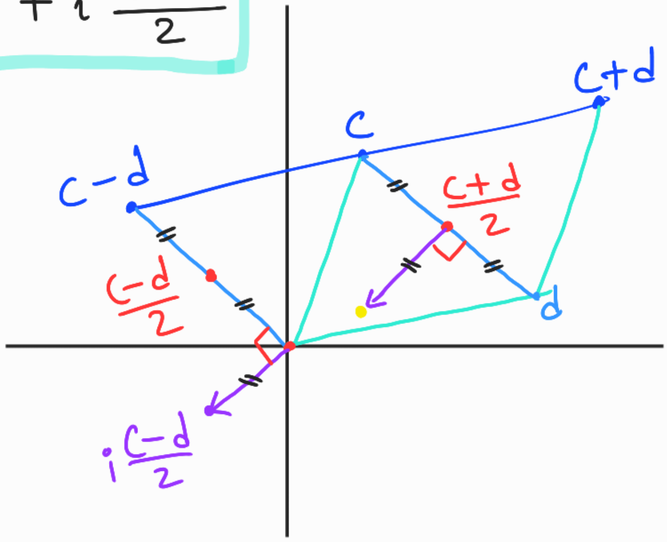
$$= \frac{d+c}{2} + i\,\frac{c-d}{2}$$

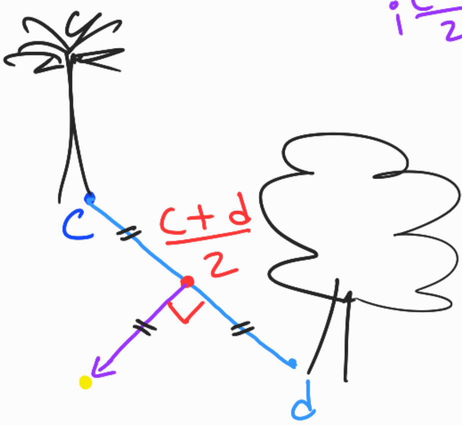
(does not depend on z)

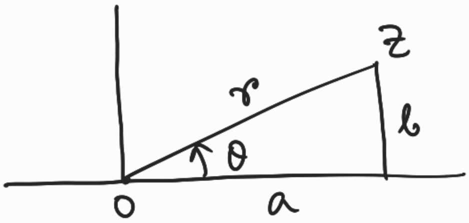
One way: Place a boulder at any point, it'll lead you to the same point t (treasure).



$$\boxed{t = \frac{d+c}{2} + i\,\frac{c-d}{2}}$$







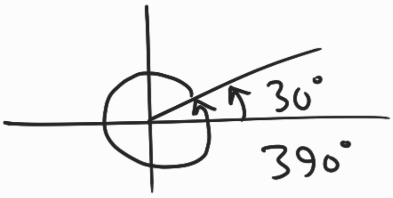
$$z = a + ib$$

$$r = |z| = \sqrt{a^2+b^2}$$

"modulus of z" distance of z from origin

$\theta = \arg z$

"argument of z" (In detail — later)



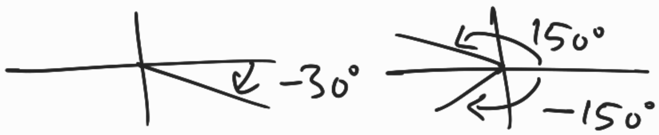
$$\{\theta,\ \theta+2\pi,\ \theta-2\pi,\ \ldots\} = \arg z$$

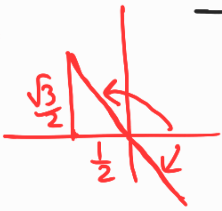
If $\theta \in (-\pi, \pi]$ then $\theta = \operatorname{Arg} z$

$(-180°, 180°]$ "Principal arg. of z"

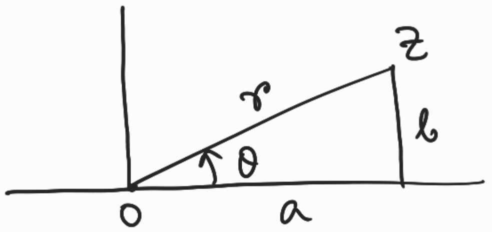
$\operatorname{Arg}(-1) = \pi$

$\operatorname{Arg}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \frac{2\pi}{3}$





$\operatorname{Arg}\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -\pi/3.$

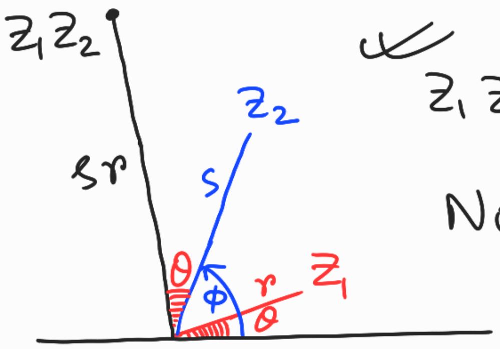


$$|z| = \sqrt{a^2+b^2}$$
$$a = r\cos\theta \qquad \text{(for any}$$
$$b = r\sin\theta \qquad \text{arg. } \theta \text{ of } z)$$

$$z = a+ib \longleftrightarrow z = r(\cos\theta + i\sin\theta).$$

$$z_1 = r(\cos\theta + i\sin\theta)$$
$$z_2 = s(\cos\phi + i\sin\phi)$$

What is $z_1 z_2$?



✓ $z_1 z_2 = rs(\cos(\theta+\phi) + i\sin(\theta+\phi))$

Now also check this algebraically.

✓ $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi)$
$+ i\sin(\theta+\phi)$

Hence,

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

More generally, for any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$,

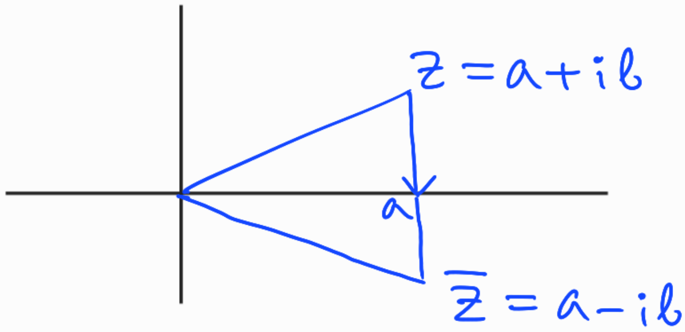
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(de Moivre's formula)

Another point: (inverse of z, or, $1/z$)

$$(\cos\theta + i\sin\theta)\underbrace{(\cos(-\theta) + i\sin(-\theta))}_{= \cos\theta - i\sin\theta} = \cos 0 + i\sin 0 = 1$$

$\therefore$ for $z = \cos\theta + i\sin\theta$, $1/z = \cos\theta - i\sin\theta$.

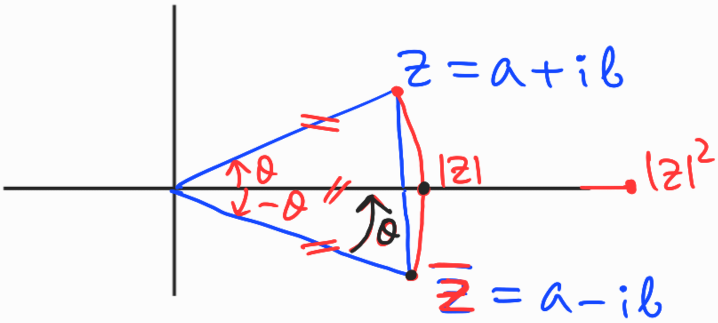


$$a = \frac{z + \bar{z}}{2}$$

$$b = \frac{z - \bar{z}}{2i}$$

$$z\bar{z} = ?$$

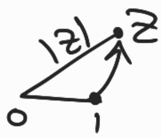
$$z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2.$$



$z z_1$ → rotate z_1 about origin by angle θ and stretch by a factor of $|z|$.

$z\bar{z}$ = rotate $\bar{z}$ by angle θ around origin, stretch by a factor of $|z|$

$= |z| \times |z| = |z|^2.$



$$\boxed{z\bar{z} = |z|^2} \longrightarrow 1/z = \frac{\bar{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}.$$

Some properties (verify)

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad |z_1 z_2| = |z_1| \cdot |z_2|$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

Triangle inequality

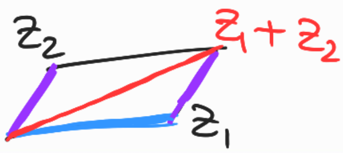
$$|z^n| = |z|^n \quad (n \in \mathbb{Z})$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|z_1/z_2| = \frac{|z_1|}{|z_2|}$$

equality iff?

(Hint: C-S)



$z_1 = a + ib \qquad z_2 = c + id$

$$\frac{z_1}{z_2} = (a+ib)\frac{1}{(c+id)} = \frac{(a+ib)(c-id)}{c^2+d^2}.$$

① Prove that, for any $z_1, z_2 \in \mathbb{C}$,

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right).$$

We know, $|z|^2 = z\bar{z}$. Hence,

$$|z_1+z_2|^2 + |z_1-z_2|^2$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

$$= 2(z_1\bar{z}_1 + z_2\bar{z}_2)$$

$$= 2\left(|z_1|^2+|z_2|^2\right).$$

x, y positive reals

② Find all $x, y > 0$ s.t.

$$\sqrt{x}\left(1+\frac{1}{x+y}\right) = \frac{3}{2}, \quad \sqrt{y}\left(1-\frac{1}{x+y}\right) = \frac{1}{2}.$$

Take $x = a^2$, $y = b^2$. Then

(i) $a + \dfrac{a}{a^2+b^2} = \dfrac{3}{2}$ (ii) $b - \dfrac{b}{a^2+b^2} = \dfrac{1}{2}$.

<u>Recall</u> $\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$

Eqn (I) + $i \times$ Eqn (II) gives

$$\underbrace{(a+ib)}_{=z\text{ (say)}} + \frac{a-ib}{a^2+b^2} = \frac{3}{2} + i\frac{1}{2}$$

$$\Updownarrow$$

$$z + \frac{1}{z} = \frac{3+i}{2}$$

$$\Longleftrightarrow \quad 2z^2 - (3+i)z + 2 = 0$$

$$\Longleftrightarrow \quad z = \frac{3+i \pm \sqrt{(3+i)^2 - 16}}{4}$$

$$= \frac{3+i \pm \sqrt{-8+6i}}{4}$$

$$= \frac{3+i \pm \sqrt{(3i+1)^2}}{4}$$

$$= \frac{(3+i) \pm (3i+1)}{4}$$

$$-8+6i = 9i^2 + 6i + 1 = (3i+1)^2$$

$\therefore\ a+ib = 1+i$, or $\frac{1}{2} - \frac{1}{2}i$.

$\Longrightarrow (x, y) = (a^2, b^2) = (1, 1)$ or $\left(\frac{1}{4}, \frac{1}{4}\right)$.

$\left(\frac{1}{4}, \frac{1}{4}\right)$ does not satisfy $\sqrt{y}\left(1 - \frac{1}{x+y}\right) = \frac{1}{2}$.

$\therefore\ (x, y) = (1, 1)$ is the only possible solution.

Ref. Complex Numbers from A to...Z by Titu Andreescu.