05/06/21

(Sheet-1)

#10 Let a, b, c be non-zero complex no.s having the same modulus. If $az^2 + bz + c = 0$ has a root with modulus 1, then show that $b^2 = ac$.

Sol Let α, β be the two roots of the equation $az^2 + bz + c = 0$. Then

$$\alpha + \beta = -b/a, \quad \alpha\beta = c/a.$$

$|\alpha\beta| = \frac{|c|}{|a|} = 1$ $\therefore$ Since one among α, β has mod. 1 the other one will also have mod. 1.

$$[\,|\alpha| = 1 \iff |\beta| = 1\,]$$

$\therefore |\alpha| = 1 = |\beta|$.

$$\alpha + \beta = -\frac{b}{a} \Rightarrow |\alpha + \beta| = \frac{|b|}{|a|} = 1$$

$$\Rightarrow 1 = |\alpha + \beta|^2 = (\alpha + \beta)\overline{(\alpha + \beta)}$$

$$= (\alpha + \beta)(\bar{\alpha} + \bar{\beta})$$

$$= (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$\alpha\bar{\alpha} = \beta\bar{\beta} = 1$
$\Rightarrow \bar{\alpha} = 1/\alpha$
$\bar{\beta} = 1/\beta$

$$\Rightarrow \alpha\beta = (\alpha + \beta)^2$$

$$\Rightarrow \frac{c}{a} = \left(-\frac{b}{a}\right)^2 \Rightarrow b^2 = ac.$$

#11 Applying the same idea as in #10, we get

$$b^2 = ac \quad \text{and} \quad c^2 = ab.$$

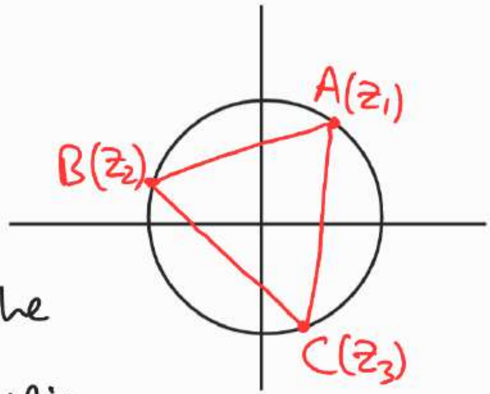
Q. $|a| = |b| = |c|$ and $b^2 = ac$, $c^2 = ab$. Then Show that a, b, c are vertices of an equilateral triangle. [We'll discuss this later.]

<u>#2'</u> $z_1, z_2, z_3 \in \mathbb{C}$ s.t. $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$. Prove that

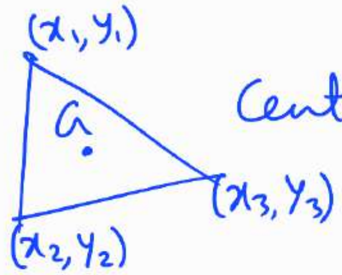
(i) z_1, z_2, z_3 are vertices of an equilateral triangle.

(ii) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$.

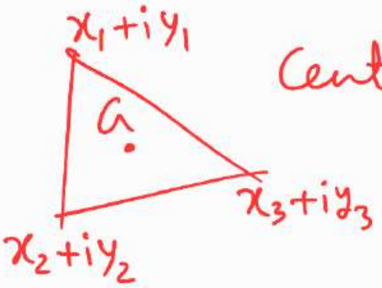
<u>Sol^n</u> $|z_1| = |z_2| = |z_3| = 1$ means z_1, z_2, z_3 lies on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.



Denoting by A, B, C the points z_1, z_2, z_3, we can say that circumcircle of ΔABC is the unit circle centred at the origin.



Centroid $G = \left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$



$$\text{Centroid } G = \frac{x_1 + x_2 + x_3}{3} + i\,\frac{y_1 + y_2 + y_3}{3}$$

$$= \frac{1}{3}\Big((x_1 + iy_1) + (x_2 + iy_2) + (x_3 + iy_3)\Big).$$

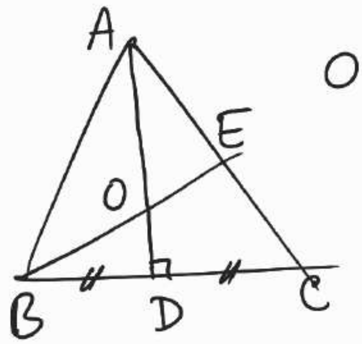
$\therefore$ Centroid of $\triangle ABC$ is

$$G = \frac{(z_1 + z_2 + z_3)}{3} = 0.$$

$\therefore$ Centroid $\equiv$ circumcentre for $\triangle ABC$

$$\Downarrow$$

$\triangle ABC$ is equilateral. Why?



$O \rightarrow$ centroid as well as circumcentre

$OA = OB = OC$ $\because$ circumcentre

and AO bisects BC, BO bisects CA ($\because$ centroid).

$\triangle ABD \cong \triangle ACD$ [$\because$ $BD = DC$,

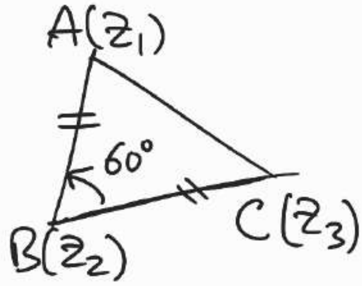
AD common,

($\because$ O circumcentre, D midpt of BC) $\longrightarrow$ $\angle ADB = \angle ADC = 90^\circ$]

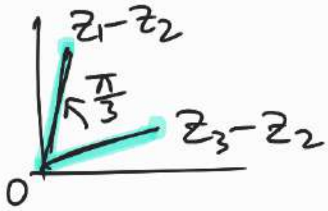
$\therefore$ $\triangle ABD \cong \triangle ACD$ (SAS) $\Rightarrow$ $AB = AC$.

Similarly we can prove that $BC = AB$.

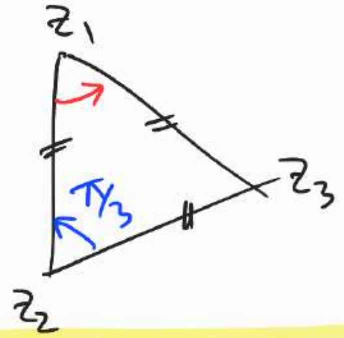
$\therefore$ $\triangle ABC$ is equilateral.

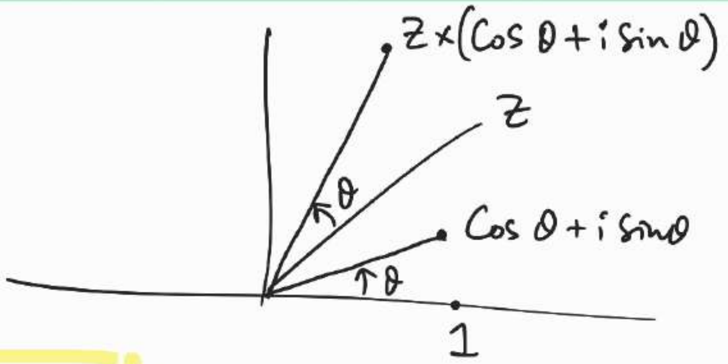


$$(z_3 - z_2)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = (z_1 - z_2)$$



[from rotation formula]





$$\boxed{\begin{aligned}(z_1 - z_2) &= (z_3 - z_2)\alpha \\ (z_3 - z_1) &= (z_2 - z_1)\alpha \\ &= (z_3 - z_2)(-\alpha^2)\end{aligned}}$$

$$\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\alpha^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$\alpha^3 = \cos\pi + i\sin\pi = -1.$$

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1$$

$$= \frac{1}{2}\left[(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2\right]$$

$$= \frac{1}{2}(z_2 - z_3)^2\left[\alpha^2 + 1 + \alpha^4\right]$$

$$= \frac{1}{2}(z_2 - z_3)^2(1 + \alpha^2 + \alpha^4)$$

$$= \frac{1}{2}(z_2 - z_3)^2\,\frac{\alpha^6 - 1}{\alpha^2 - 1}$$

$$= 0 \quad \left(\because \alpha^6 = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = 1\right.$$

$$\left.\text{and } \alpha^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \neq 1\right)$$

$$\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\alpha^3 = -1, \ \alpha^6 = 1$$

$$-\alpha^2 = \alpha^5 = \frac{1}{\alpha} = \overline{\alpha}$$

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$z_1 + z_2 + z_3 = 0$. Hence show that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = 0.$$

de Moivre's thm

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Q. Is it also the case that

$$(\cos\theta + i\sin\theta)^{p/q} = \cos \tfrac{p}{q}\theta + i\sin\tfrac{p}{q}\theta \,?$$

$\left(p/q = 1/3,\ \theta = 2\pi\right)$

$$\underset{(\text{say})}{\omega} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = (\cos 2\pi + i\sin 2\pi)^{1/3}$$

$$= 1^{1/3}$$

$\left(p/q = 1/3,\ \theta = 0\right)$

$$1 = \cos 0 + i\sin 0 = (\cos 0 + i\sin 0)^{1/3}$$

$$= 1^{1/3}$$

But $\omega \neq 1$. Then?

$$1^{1/3} = z \iff z^3 = 1$$

$$\iff (z-1)(z^2+z+1) = 0$$

$$\iff z = 1,\ -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}$$

$$\iff z = 1, \omega, \omega^2.$$

[ω can be either $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ or $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$]

$\therefore$ $1^{1/3}$ is not a single complex number.

$$1^{1/3} = \{1, \omega, \omega^2\} = \{1, -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}\}$$

$$\begin{array}{l|l} \omega = 1^{1/3} & \text{interpret as} \rightarrow \quad \omega \in 1^{1/3} \\ 1 = 1^{1/3} & \quad\quad\quad\quad\quad\quad\quad\;\; 1 \in 1^{1/3} \end{array}$$

In general, $(\cos\theta + i\sin\theta)^{p/q}$ has more than one value, and

$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of them.

How?

$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q$$

$$= \cos p\theta + i\sin p\theta \quad \text{(by de Moivre)}$$

$$= (\cos\theta + i\sin\theta)^p = z \quad (\quad " \quad)$$

$\therefore \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.
$= v$

$v^q = z \Rightarrow v \in z^{1/q}$

Find all values of $z^{1/n}$

Q. Given z, find all α s.t. $\alpha^n = z$ $(n \in \mathbb{N})$.

[Then, all values of $(\cos\theta + i\sin\theta)^{p/q}$ will be all the qth roots of $(\cos p\theta + i\sin p\theta)$.]

Q. Find all cubic roots of (-8).

$$z^3 = -8$$

$$\Rightarrow (z/-2)^3 = 1$$

$$\Rightarrow z/-2 = 1, \omega \text{ or } \omega^2$$

$$\Rightarrow z = -2, -2\omega \text{ or } -2\omega^2.$$

Q. Find all values of $z^{1/n}$. $\longrightarrow$ enough to solve this for $z=1$

$\because$ If $1^{1/n} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ (n^{th} roots of unity/1)

and if z_0 is one of the values of $z^{1/n}$

then all values of $z^{1/n}$ will be $\{z_0\varepsilon_1, \ldots, z_0\varepsilon_n\}$.

$$\alpha^n = z \quad \& \quad z_0^n = z$$

$$\Leftrightarrow (\alpha/z_0)^n = 1 \Leftrightarrow \alpha/z_0 \in \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$$

$$\Leftrightarrow \alpha \in \{z_0\varepsilon_1, z_0\varepsilon_2, \ldots, z_0\varepsilon_n\}.$$

Next, let us find the n^{th} roots of unity.

Goal: find all $z \in \mathbb{C}$ s.t. $z^n = 1$.

$$z^n = 1 \Rightarrow |z|^n = 1 \Rightarrow |z| = 1.$$

We can take $z = \cos\theta + i\sin\theta$.

$$z = \cos\theta + i\sin\theta.$$

$$z^n = \cos n\theta + i\sin n\theta = 1 + 0i$$

$$\therefore \cos(n\theta) = 1, \quad \sin(n\theta) = 0$$

$$\iff n\theta = 2k\pi, \ k \in \mathbb{Z}$$

Thus, $z^n = 1$ if and only if

$$z \in \left\{ \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \mathbb{Z} \right\}$$

$$\cos(2k\pi + \theta) = \cos\theta, \quad \sin(2k\pi + \theta) = \sin\theta$$

So, enough to consider all $k \in \mathbb{Z}$ such that

$$0 \le \frac{2k\pi}{n} < 2\pi \iff 0 \le k < n$$

$$\iff k = 0, 1, 2, \ldots, n-1.$$

$\therefore$ All the n^{th} roots of 1 are

$$\left\{ \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k = 0, 1, \ldots, n-1 \right\}.$$