Complex Numbers

12/06/21

✯ Roots of unity:

The n^{th} roots of unity are basically the n complex roots of the equation $z^n = 1$.

We saw in the last class that the n^{th} roots of unity are

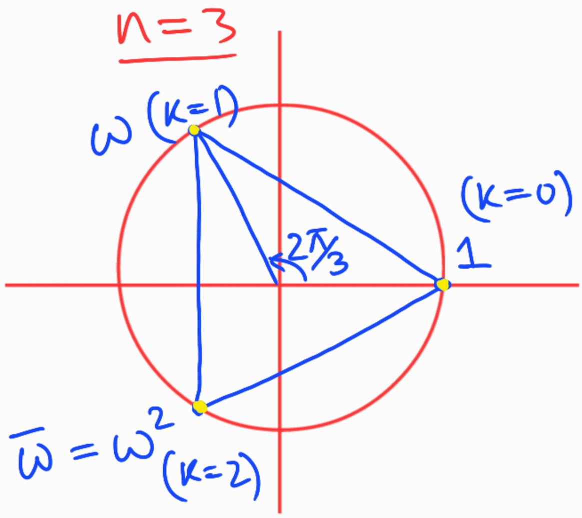
$$\left\{ \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} : k = 0, 1, \ldots, n-1 \right\}$$

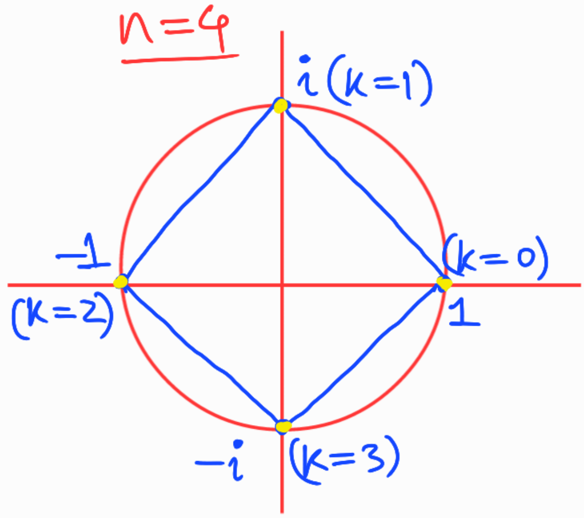
Call $\epsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Then

$$\epsilon_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right) = \epsilon^k \quad \text{(de Moivre)}$$

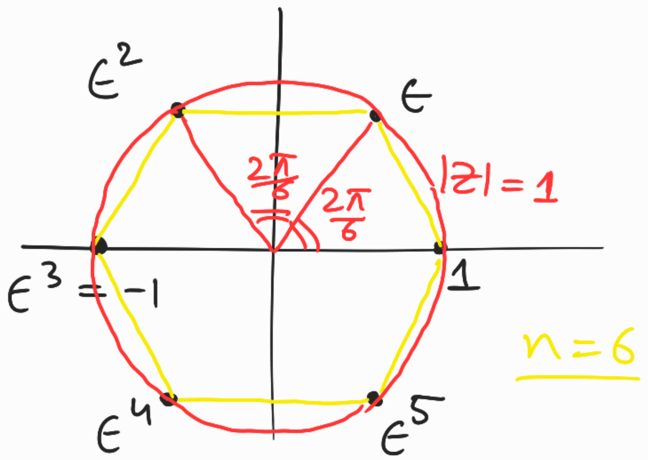
$\therefore$ The n^{th} roots of unity are $1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$

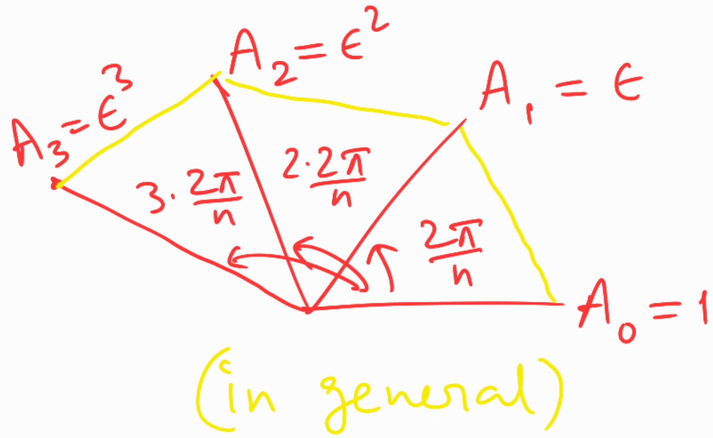
where $\epsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$.

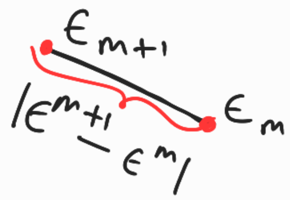




The n roots form the vertices of a regular Polygon.







A_m $\epsilon_m = \cos \frac{2\pi m}{n} + i \sin \frac{2\pi m}{n}$

$$\overline{A_m A_{m+1}} = |\epsilon^{m+1} - \epsilon^m| = |\epsilon^m| \cdot |\epsilon - 1|$$

$$= |\epsilon - 1| \quad (\because |\epsilon| = 1)$$

$\therefore$ $\overline{A_m A_{m+1}}$ is same for each $0 \leq m \leq n-1$.

① Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the n^{th} roots of unity.

Find a) $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$

b) $\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \cdots + \varepsilon_{n-1} \varepsilon_n$

c) $\varepsilon_1^k + \varepsilon_2^k + \cdots + \varepsilon_n^k$ for $1 \leq k \leq n-1$

d) $\varepsilon_1^k + \varepsilon_2^k + \cdots + \varepsilon_n^k$ in general ($k \in \mathbb{N}$).

Sol^n Since $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are roots of the poly.

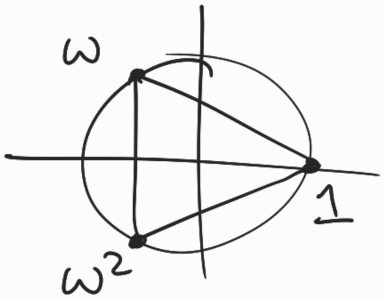
$P(z) = z^n - 1$, we can apply Vieta's thm

to say that a) $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n = 0$,

and b) $\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \ldots + \varepsilon_{n-1}\varepsilon_n = 0$.

b') $\varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots \varepsilon_n = (-1)^n \frac{(-1)}{1} = (-1)^{n+1}$.

Intuition: $\omega^2 = \bar{\omega} = \frac{1}{\omega}$

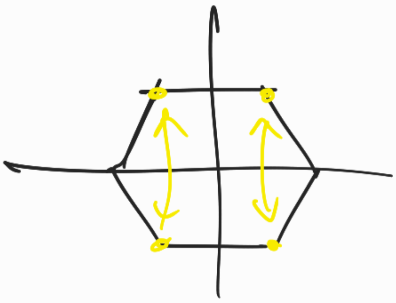


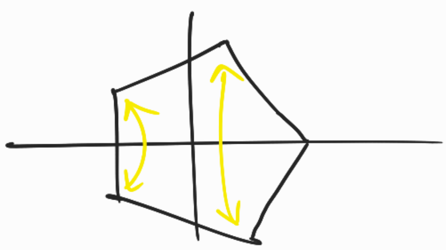
If n is odd, we can pair up ε and $1/\varepsilon = \bar{\varepsilon}$, and their product is 1, so prod. of roots $= 1 \; \varepsilon_1 \bar{\varepsilon}_1 \varepsilon_2 \bar{\varepsilon}_2 \cdots = 1$.

If n is even,

the real roots are 1 and -1, while the complex roots pair up as ε_j and $\bar{\varepsilon}_j = \frac{1}{\varepsilon_j}$.

That's why prod. of roots $= (-1)$ in this case.





Another idea

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = \varepsilon^{0+1+\dots+(n-1)} = \varepsilon^{\frac{(n-1)n}{2}}$$

$n = 2k$ $\qquad \varepsilon^{\frac{(n-1)n}{2}} = (\varepsilon^k)^{(2k-1)} \overset{*}{=} (-1)^{2k-1} = -1$

$$* \quad \varepsilon^k = \left(\cos\frac{2\pi}{2k} + i\sin\frac{2\pi}{2k}\right)^k = \cos\pi + i\sin\pi = -1.$$

$n = 2k+1$ $\qquad \varepsilon^{\frac{(n-1)n}{2}} = (\varepsilon^{2k+1})^k = 1^k = 1$ $\qquad \varepsilon^n = 1$

c) Let $\epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then

$$\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} = \{1, \epsilon, \epsilon^2, \dots, \epsilon^{n-1}\}.$$

$$\varepsilon_1^k + \varepsilon_2^k + \dots + \varepsilon_n^k$$

$$= 1 + \epsilon^k + \epsilon^{2k} + \dots + \epsilon^{(n-1)k}$$

$$= 1 + r + r^2 + \dots + r^{n-1} \qquad (r = \epsilon^k)$$

Fact (Sum of a G.P.)

$$1 + r + r^2 + \dots + r^{n-1} = \begin{cases} \dfrac{r^n - 1}{r - 1} & \text{if } r \neq 1 \\ n & \text{if } r = 1 \end{cases}$$

Then, $\varepsilon_1^k + \varepsilon_2^k + \dots + \varepsilon_n^k$

$$= 1 + r + r^2 + \dots + r^{n-1} \quad (r = \epsilon^k)$$

$$= \frac{r^n - 1}{r - 1} \quad \left[\because r - 1 = \epsilon^k - 1 \neq 0 \text{ for } 1 \leq k \leq n-1\right]$$

$\left(r - 1 = \epsilon^k - 1 = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} - 1 \neq 0\right.$

because $0 < \theta = \frac{2k\pi}{n} < 2\pi$, so we can

not have $\cos\theta = 1$, $\sin\theta = 0$ simultaneously.$\left.\right)$

$$= \frac{(\epsilon^k)^n - 1}{\epsilon^k - 1} = \frac{1^k - 1}{\epsilon^k - 1} = 0. \quad (\because \epsilon^n = 1)$$

(Ans)

d) $\varepsilon_1^k + \varepsilon_2^k + \dots + \varepsilon_n^k$

$$= 1 + r + r^2 + \dots + r^{n-1} \quad (r = \epsilon^k)$$

$$= \begin{cases} \frac{r^n - 1}{r - 1} = 0 & \text{when } \epsilon^k \neq 1 \\ n & \text{when } \epsilon^k = 1 \end{cases}$$

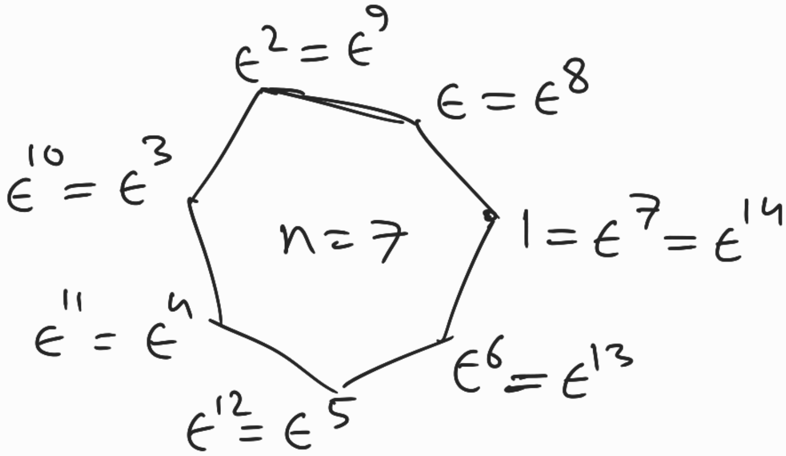
When?

$$\epsilon^k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 1$$

$$\iff \frac{2k\pi}{n} = 2m\pi \text{ for some integer } m$$

$$\iff k = mn \text{ for some integer } m$$

$$\iff n \mid k.$$



$$\therefore \; \varepsilon_1^k + \dots + \varepsilon_n^k = \begin{cases} 0 & \text{if } n \nmid k, \\ n & \text{if } n \mid k. \end{cases}$$

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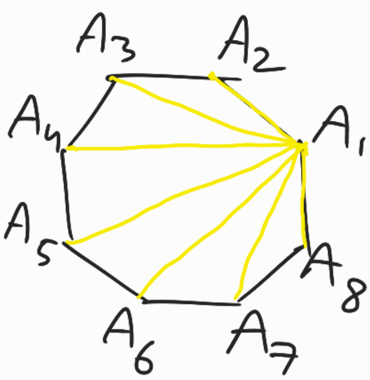
② Suppose that $A_1 A_2 \cdots A_n$ is a regular n-gon inscribed in a circle of radius 1.

Find $\overline{A_1A_2}\;\overline{A_1A_3} \cdots \overline{A_1A_n}$.



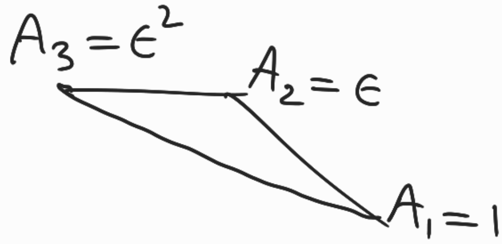
③ Simplify:

$$\sin\frac{\pi}{2n}\,\sin\frac{3\pi}{2n} \cdots \sin\frac{(2n-1)\pi}{2n}.$$

Sol^n to ②

We set up a complex coord. system s.t. $A_1, A_2, \ldots, A_n$ have the complex coordinates $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}$ (the n^{th} roots of unity) where

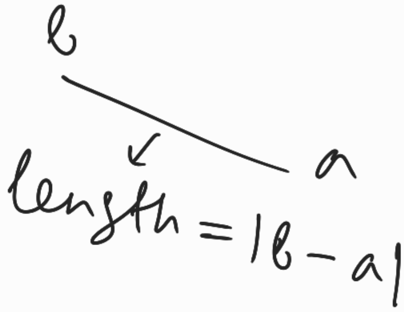
$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$



Then

$$\overline{A_1A_2} = |\epsilon - 1|$$

$$\overline{A_1A_3} = |\epsilon^2 - 1|$$



and so on.

$$\overline{A_1A_2} \times \overline{A_1A_3} \times \cdots \times \overline{A_1A_n}$$

$$= |\epsilon - 1| \times |\epsilon^2 - 1| \times \cdots \times |\epsilon^{n-1} - 1|$$

$$= |(\epsilon - 1)(\epsilon^2 - 1)\cdots(\epsilon^{n-1} - 1)|$$

$1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1} \longrightarrow n^{th}$ roots of unity

i.e., roots of $z^n - 1 = 0$.

Hint $\epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$ roots of which poly.?

$$1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1} \longleftarrow \underbrace{z^n - 1} = 0$$

$$\underline{(z-1)}\,\underbrace{(z^{n-1} + \ldots + z + 1)} = 0$$

roots: $\epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$

$\epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$ are roots of the polynomial

$$P(z) = z^{n-1} + z^{n-2} + \ldots + z + 1.$$

So,

$$P(z) = (z-\epsilon)(z-\epsilon^2)\cdots(z-\epsilon^{n-1}).$$

$$\therefore\ (z-\epsilon)(z-\epsilon^2)\cdots(z-\epsilon^{n-1}) = z^{n-1} + z^{n-2} + \ldots + z + 1.$$
$$= P(z)$$

Hence,

$$\overline{A_1A_2}\ \overline{A_1A_3} \cdots \overline{A_1A_n}$$

$$= \left|(1-\epsilon)(1-\epsilon^2)\cdots(1-\epsilon^{n-1})\right|$$

$$= |P(1)| = n. \quad \text{(Ans)}$$

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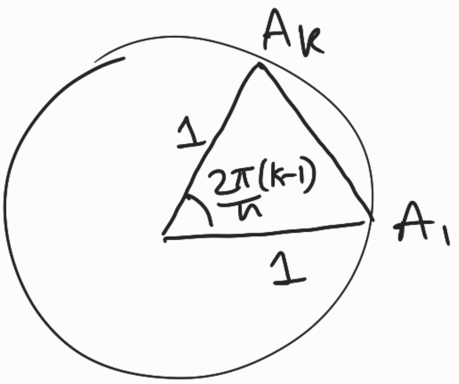
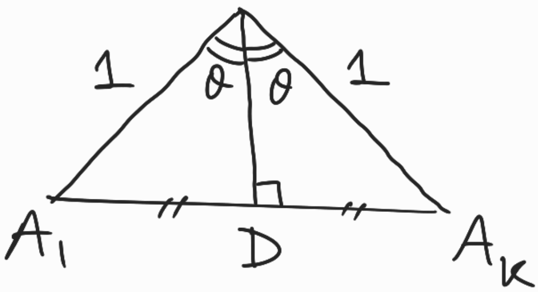
③ Simplify:

$$\sin\frac{\pi}{2n}\ \sin\frac{3\pi}{2n} \cdots \sin\frac{(2n-1)\pi}{2n}.$$





$$2\theta = \frac{2\pi(k-1)}{n}$$

$$\theta = \frac{(k-1)\pi}{n}$$

$$\overline{A_1D} = \overline{DA_k} = \sin\theta$$

$$\therefore \boxed{\overline{A_1A_k} = 2\sin\frac{(k-1)\pi}{n}}$$

Now,

$$\overline{A_1A_2}\ \overline{A_1A_3} \cdots \overline{A_1A_n} = n$$ (from prev. Problem)

$$\Rightarrow 2\sin\frac{\pi}{n}\ 2\sin\frac{2\pi}{n} \cdots 2\sin\frac{(n-1)\pi}{n} = n$$

$$\Rightarrow \boxed{\sin\frac{\pi}{n}\ \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}} \cdots (1)$$

Since this holds for every $n \in \mathbb{N}$, we can use it for $2n$ (in place of n) and write

$$\sin\frac{\pi}{2n}\ \sin\frac{2\pi}{2n} \cdots \sin\frac{(2n-1)\pi}{2n} = \frac{2n}{2^{2n-1}} \cdots (11)$$

$n=8$ $$\sin\frac{\pi}{8}\ \cancel{\sin\frac{2\pi}{8}}\ \sin\frac{3\pi}{8}\ \cancel{\sin\frac{4\pi}{8}} \cdots \cancel{\sin\frac{6\pi}{8}}\ \sin\frac{7\pi}{8}$$

$n=4$ $$\cancel{\sin\frac{\pi}{4}} \cdot \cancel{\sin\frac{2\pi}{4}} \cdot \cancel{\sin\frac{3\pi}{4}}$$

Dividing eqn. (11) by eqn. (1), we get

$$\sin\frac{\pi}{2n}\,\sin\frac{3\pi}{2n}\cdots\sin\frac{(2n-1)\pi}{2n}$$

$$= \frac{2n/2^{2n-1}}{n/2^{n-1}} = \frac{1}{2^{n-1}}. \quad \text{(Ans)}$$

<u>Examples</u>

- $\sin\frac{\pi}{8}\,\sin\frac{3\pi}{8}\,\sin\frac{5\pi}{8}\,\sin\frac{7\pi}{8} = \frac{1}{8}.$
- $\sin\frac{\pi}{7}\,\sin\frac{2\pi}{7}\cdots\sin\frac{6\pi}{7} = \frac{7}{2^6}$

$\sin\frac{\pi}{7} = \sin\frac{6\pi}{7}$ etc. So,

$\therefore \sin\frac{\pi}{7}\,\sin\frac{2\pi}{7}\,\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}.$