19/06/21

① Suppose that a, b, c are complex numbers such that the roots of the equation

$$x^3 + ax^2 + bx + c = 0$$

have the same modulus. Then show that

$$a = 0 \iff b = 0.$$

Sol^n Let α, β, γ be the roots. It is given that

$$|\alpha| = |\beta| = |\gamma| = r \text{ (say)}.$$

We also know from Vieta's formulae that

$$\alpha + \beta + \gamma = -a, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = b,$$

and $\alpha\beta\gamma = -c.$

How to relate them?

To show: $a = 0$ if and only if $b = 0$.

Let's start with $a = 0$.

$$a = 0 \implies \alpha + \beta + \gamma = 0$$

How?

$$\implies \alpha\beta + \beta\gamma + \gamma\alpha = 0 \implies b = 0$$

Let us assume first that $\underline{\underline{r \neq 0.}}$ Then $\underline{\underline{\alpha, \beta, \gamma \neq 0.}}$

$$a = 0 \iff \alpha + \beta + \gamma = 0$$

$$\iff \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0 \qquad \alpha\bar{\alpha} = |\alpha|^2 = r^2$$

$$\bar{\alpha} = r^2/\alpha \text{ etc.}$$

$$\iff \frac{r^2}{\alpha} + \frac{r^2}{\beta} + \frac{r^2}{\gamma} = 0$$

$$\iff \frac{r^2}{\alpha\beta\gamma}(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$$

$$\iff \alpha\beta + \beta\gamma + \gamma\alpha = 0 \iff b = 0.$$

Finally, if $r = 0$, then $\alpha = \beta = \gamma = 0$ and the desired conclusion follows trivially:

$$-a = \alpha + \beta + \gamma = 0 = \alpha\beta + \beta\gamma + \gamma\alpha = b.$$

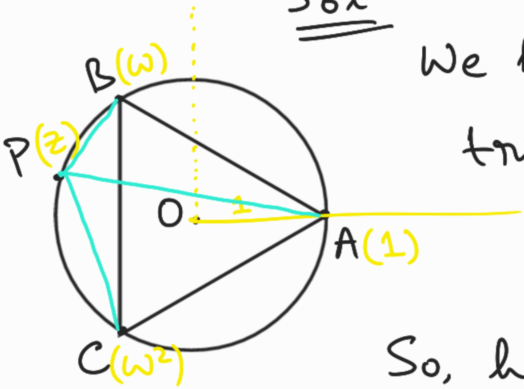
This completes the proof. $\square$

(2) $\triangle ABC$ be an equilateral triangle whose circum-circle S has radius 1. Show that for any point P on S, it holds that $PA^2 + PB^2 + PC^2 = 6$.

Want to involve Complex no.s.

— Let's try $1, \omega, \omega^2$. (Cube roots of unity)

Sol^n



We know that $1, \omega, \omega^2$ form an equilateral triangle which is inscribed in the unit circle centred at the origin.

So, here we can set up a complex coord. system such that the complex coordinates of A, B, C are nothing but $1, \omega, \omega^2$, respectively.

Let z be the complex coord. of P (which lies on the unit circle). Then,

$$PA^2 + PB^2 + PC^2 = |z-1|^2 + |z-\omega|^2 + |z-\omega^2|^2$$

$$= (z-1)(\bar{z}-1) + (z-\omega)(\bar{z}-\bar{\omega}) + (z-\omega^2)(\bar{z}-\overline{\omega^2})$$

$z\bar{z} = |z|^2 = 1$ ($\because$ P lies on the unit circle)

$\omega^3 = 1, \quad |\omega| = 1 \quad \therefore \bar{\omega} = \frac{1}{\omega} = \omega^2, \quad \overline{\omega^2} = \bar{\bar{\omega}} = \omega.$

$1 + \omega + \omega^2 = 0.$

$$= 3z\bar{z} + 1 + \omega\bar{\omega} + \omega^2\overline{\omega^2}$$

$$- \bar{z}\underbrace{(1+\omega+\omega^2)}_{=0} - z\underbrace{(1+\bar{\omega}+\overline{\omega^2})}_{=1+\omega^2+\omega=0}$$

$$= 3 + 1 \times 3$$

$$= 6$$

as required.

③ Find the value of

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \dots + \cos\frac{9\pi}{11}.$$

<u>Sol^n</u> Let $z = \cos\frac{\pi}{11} + i\sin\frac{\pi}{11}$. Then

$$z^{11} = \left(\cos\frac{\pi}{11} + i\sin\frac{\pi}{11}\right)^{11} = \cos\pi + i\sin\pi = -1$$

by de Moivre's theorem.

$w = a+ib$ Then $a = \mathrm{Re}(w) = \dfrac{w+\bar{w}}{2}$

$\bar{w} = a - ib$ $\quad b = \mathrm{Im}(w) = \dfrac{w-\bar{w}}{2i}$

$\mathrm{Re}(w_1 + w_2) = \mathrm{Re}(w_1) + \mathrm{Re}(w_2)$. $\longleftarrow$ Obviously!

$z = \cos\frac{\pi}{11} + i\sin\frac{\pi}{11}$. Then we can write

$$\cos\frac{k\pi}{11} = \mathrm{Re}\left[\cos\frac{k\pi}{11} + i\sin\frac{k\pi}{11}\right] \underset{\uparrow}{=} \mathrm{Re}(z^k).$$

(by de Moivre's thm)

Hence

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \dots + \cos\frac{9\pi}{11}$$

$$= \mathrm{Re}(\underline{z + z^3 + z^5 + z^7 + z^9})$$

sum of G.P.

$$= \mathrm{Re}\left(z\,(1 + z^2 + z^4 + z^6 + z^8)\right)$$

$$= \mathrm{Re}\left(z \cdot \frac{1-(z^2)^5}{1-z^2}\right) \qquad \text{[sum of G.P.]}$$

$$= \mathrm{Re}\left(z \frac{1 - z^{10}}{1 - z^2}\right) \qquad z^{11} = -1$$

$$z^{10} = -1/z$$

$$= \mathrm{Re}\left(z \frac{(1 + 1/z)}{(1 - z^2)}\right)$$

$$\boxed{|z| = 1 \Rightarrow \mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}}$$

$$= \mathrm{Re}\left(\frac{1}{1-z}\right).$$

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1-\bar{z}}\right)$$

$$= \frac{1}{2}. \quad \text{(Ans)}$$

$$= \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1-1/z}\right) = \frac{1}{2}.$$

(4) Show that

$$\left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right)^2 + \left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right)^2 = 2^n.$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

<u>Binomial theorem</u>

$$(x+y)^n = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n.$$

In particular,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \underbrace{\binom{n}{2} x^2}_{\binom{n}{2}(-1)} + \cdots + \binom{n}{n} x^n \quad \text{———} (*)$$

Putting $x = i$, we get

$$(1+i)^n = \binom{n}{0} + \binom{n}{1} i - \binom{n}{2} - \binom{n}{3} i + \binom{n}{4} - \cdots \quad \text{———} (\mathrm{I})$$

Putting $x = -i$ in $(*)$ we get

$$(1-i)^n = \binom{n}{0} - \binom{n}{1} i - \binom{n}{2} + \binom{n}{3} i + \binom{n}{4} + \dots \quad \text{--- (II)}$$

(I) + (II) gives

Re$[(1+i)^n]$

$$\frac{(1+i)^n + (1-i)^n}{2} = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \dots = a \text{ (say).}$$

(I) − (II) gives

Im$[(1+i)^n]$

$$\frac{(1+i)^n - (1-i)^n}{2i} = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \dots = b \text{ (say).}$$

Hence,

$$a^2 + b^2 = \mathrm{Re}((1+i)^n)^2 + \mathrm{Im}((1+i)^n)^2$$

$$= |(1+i)^n|^2$$

$$= |1+i|^{2n}$$

$$= \sqrt{2}^{2n} = 2^n.$$

$\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 = |z|^2$

$a^2 + b^2 = |z|^2$

$z = a + ib$ definition!

This gives us the desired identity.

Alt. $a^2 + b^2 = \left(\frac{(1+i)^n + (1-i)^n}{2}\right)^2 - \left(\frac{(1+i)^n - (1-i)^n}{2}\right)^2$

$= (1+i)^n (1-i)^n$

$= (1-i^2)^n = 2^n.$

$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy$