Solution to Class Test on Complex Numbers

March 2020

1. Suppose that the roots of the equation $x^4 + ax^3 + bx^2 + ax + c = 0$ are $\alpha_1, \alpha_2, \alpha_3$, and α_4. Show that, $(\alpha_1^2 + 1)(\alpha_2^2 + 1)(\alpha_3^2 + 1)(\alpha_4^2 + 1) = (1 - b + c)^2$.

   Solution. Since $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of this equation, we must have

   $$p(x) = x^4 + ax^3 + bx^2 + ax + c = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4). \tag{1}$$

   On the other hand, $\prod_{k=1}^{4}(\alpha_k^2 + 1) = \prod_{k=1}^{4}|\alpha_k + i|^2 = \prod_{k=1}^{4}(\alpha_k + i)(\alpha_k - i)$. Now from equation (1) we have $p(i)p(-i) = \prod_{k=1}^{4}(i - \alpha_k)(-i - \alpha_k)$, which is exactly the right hand side of the last equation. Hence

   $$\prod_{k=1}^{4}(\alpha_k^2 + 1) = p(i)p(-i) = \cdots = (1 - b + c)^2.$$

2. Let $\triangle ABC$ be an equilateral triangle with the circumradius equal to 1. Prove that for any point P on the circumcircle, we have $PA^2 + PB^2 + PC^2 = 6$.

   Solution. Let us set up a complex coordinate system such that the circumcircle of $\triangle ABC$ is the unit circle centred at the origin and the complex coordinate of A is 1. The coordinates of B and C can be taken as ω and $\omega^2 = \overline{\omega}$, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Let z be the coordinate of point P, which satisfies $|z| = 1$ (as P lies on the circumcirle). Now,

   $$\begin{aligned} PA^2 + PB^2 + PC^2 &= |z - 1|^2 + |z - \omega|^2 + |z - \overline{\omega}|^2 \\ &= (z - 1)(\overline{z} - 1) + (z - \omega)(\overline{z} - \overline{\omega}) + (z - \overline{\omega})(\overline{z} - \omega) \\ &= 3|z|^2 + 3 - (z + \overline{z})(1 + \omega + \omega^2) = 6. \end{aligned}$$

3. Determine the value of $\cos\dfrac{\pi}{11} + \cos\dfrac{3\pi}{11} + \cos\dfrac{5\pi}{11} + \cos\dfrac{7\pi}{11} + \cos\dfrac{9\pi}{11}$.

   Solution. Let $z = \cos\dfrac{\pi}{11} + i\sin\dfrac{\pi}{11}$. In light of de Moivre's theorem, observe that the given sum is just the real part of the complex number $z + z^3 + z^5 + z^7 + z^9$, which can be simplified to

   $$z(1 + z^2 + z^4 + z^6 + z^8) = \frac{z - z^{11}}{1 - z^2} = \frac{z + 1}{1 - z^2} = \frac{1}{1 - z}.$$

The real part of this complex number equals $\frac{1}{2}\left(\frac{1}{1-z}+\frac{1}{1-\overline{z}}\right)$. Since $|z| = 1$ implies $\overline{z} = 1/z$, we get

$$\frac{1}{2}\left(\frac{1}{1-z}+\frac{1}{1-\overline{z}}\right) = \frac{1}{2}\left(\frac{1}{1-z}+\frac{1}{1-1/z}\right) = \frac{1}{2}\left(\frac{1}{1-z}+\frac{z}{z-1}\right) = \frac{1}{2}.$$

Therefore the given sum equals $\frac{1}{2}$.

4. Show that for any positive integer n, the following identity holds:

$$\left(\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\cdots\right)^2+\left(\binom{n}{1}-\binom{n}{3}+\binom{n}{5}-\cdots\right)^2 = 2^n.$$

Solution. The binomial theorem states that

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n.$$

The given quantity suggests that we should put $x = i, -i$ etc. in this equation. Indeed, if we add up such expressions for $(1+i)^n$ and $(1-i)^n$, the odd powers get cancelled out and we get

$$\frac{(1+i)^n+(1-i)^n}{2} = \binom{n}{0}+\binom{n}{2}i^2+\binom{n}{4}i^4-\cdots = \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\cdots.$$

In similar fashion, if we subtract the later from the former, we get

$$\frac{(1+i)^n-(1-i)^n}{2} = i\left(\binom{n}{1}-\binom{n}{3}+\binom{n}{5}-\cdots\right).$$

Setting $x = (1+i)^n$ and $y = (1-i)^n$, the given quantity can be simplifies as

$$\left(\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\cdots\right)^2+\left(\binom{n}{1}-\binom{n}{3}+\binom{n}{5}-\cdots\right)^2$$
$$= \left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2i}\right)^2 = \left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2 = xy.$$

Finally, $xy = (1+i)^n(1-i)^n = (1^2-i^2)^n = 2^n$.

5. An ant is moving on the coordinate plane. Initially it was at $(6,0)$. Each move of the ant consists of a counter-clockwise rotation of 60° about the origin followed

by a translation of 7 units in the positive x-direction. If the position of the ant after 2020 moves is (p, q), find the value of $p^2 + q^2$.

Solution. Let the position of the ant after n-moves be represented by a complex number z_n. Recall that multiplying a number by $\cos\theta + i\sin\theta$ rotates the object in the complex plane by an angle θ counter-clockwise. In this case, we use $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. Applying the rotation and shifting the coordinates by 7 in the positive x direction results to $z_{n+1} = wz_n + 7$ for each $n \geq 0$. Starting with $z_0 = 6$, we have $z_1 = 6w + 7$, $z_2 = w(6w + 7) + 7 = 6w^2 + 7w + 7$, and so on. Continuing in this manner, we get

$$z_{2020} = 6w^{2020} + 7w^{2019} + 7w^{2018} + \ldots + 7.$$

From de Moivre's theorem, $w^3 = -1$. Hence $w^{2019} = (w^3)^{\text{odd}} = -1 \Rightarrow w^{2020} = -w$. Now the above sum can be simplified in different ways:

<u>*Way 1*</u>

$$6w^{2020} + 7w^{2019} + 7w^{2018} + \ldots + 7 = -w^{2020} + 7\frac{(1 - w^{2021})}{1 - w} = w + 7\frac{1 + w^2}{1 - w}$$
$$= w + 7\frac{1 - 1/w}{1 - w} \ \text{(since } w^3 = -1) \ = w - 7/w = w - 7\,\overline{w}.$$

<u>*Way 2*</u>

$$\begin{aligned}
&6w^{2020} + 7w^{2019} + 7w^{2018} + \ldots + 7 \\
&= -w^{2020} + 7(1 + w + w^2 + \cdots + w^5)(w^{2015} + w^{2009} + \cdots) + \cdots \text{(keep adjusting)} \\
&= -w^{2020} + 7(1 + w + w^2 + \cdots + w^5)(w^{2015} + w^{2009} + \cdots + w^5 + w^{-1}) - 7w^{-1} \\
&= -(-w) - 7/w \quad \text{(since } (1 + w + w^2 + \cdots + w^5) = (1 - w^6)/(1 - w) = 0) \\
&= w - 7\,\overline{w} \quad \text{(as } \overline{w} = 1/w).
\end{aligned}$$

To obtain the final answer, note that $p^2 + q^2 = |w - 7\,\overline{w}|^2 = |-3 + 4\sqrt{3}i|^2 = 57$.