Solⁿs to Class Test on Complex Num.

① Let $z_1 = \cos A + i \sin A$,

$$z_2 = \cos B + i \sin B,$$

$$z_3 = \cos C + i \sin C.$$

Call the given two equations as equations (I) and (II). Then (I) $+ i \times$ (II) gives

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1,$$

which tells us that z_1, z_2, z_3 are the vertices of an equilateral triangle. Since $|z_1| = |z_2| = |z_3| = 1$, the origin is its circumcentre and hence also the centroid G. But

$$G = \frac{z_1 + z_2 + z_3}{3} = \frac{\Sigma \cos A}{3} + i \frac{\Sigma \sin A}{3}.$$

So, $\frac{\sum \cos A}{3} = 0 = \frac{\sum \sin A}{3}$, as required.

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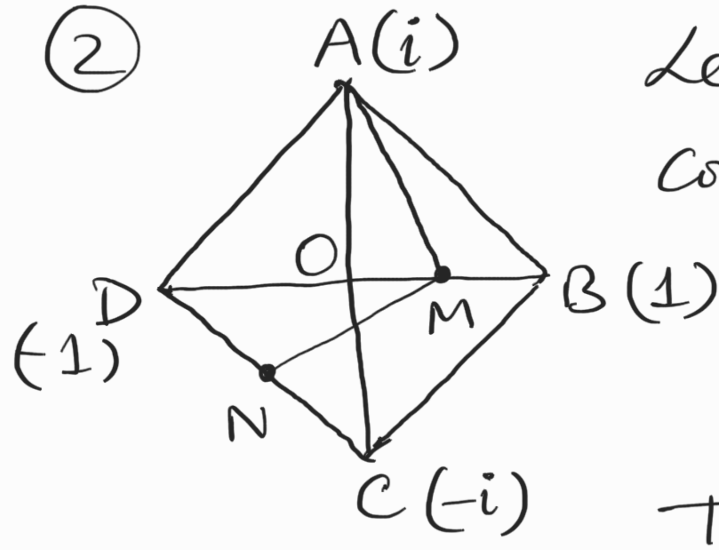
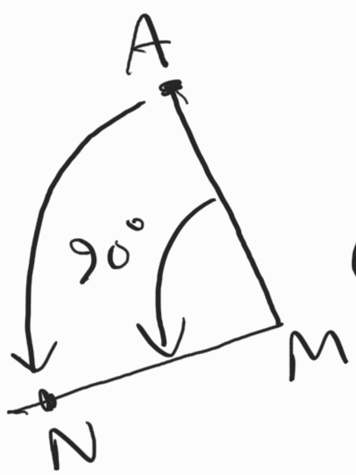
(2)



Let 1, i be the complex coordinates of A and B, and O be the origin.

Then $M \to \frac{1}{2}$

and $N \to \frac{C+D}{2} = -\left(\frac{1+i}{2}\right)$.



$$N - M = i\,(A - M)$$

Note that

$$N - M = -\frac{1+i}{2} - \frac{1}{2}$$

$$= i\left(i - \frac{1}{2}\right) = i\,(A - M).$$

Hence $\triangle AMN$ is isosceles right.

③ $A^2+B^2+C^2-AB-BC-CA$

$$= (A+B\omega+C\omega^2)(A+B\omega^2+C\omega)$$

where $\omega \neq 1$ is a complex cube root of 1.

verify this.

Here,

$A+B\omega+C\omega^2$

$$= \binom{n}{0} - \binom{n}{1}\omega + \binom{n}{2}\omega^2 - \binom{n}{3}\omega^3 + \binom{n}{4}\omega^4 - \binom{n}{5}\omega^5 + \dots$$

$$= (1-\omega)^n$$

$$A+B\omega+C\omega^2 = (1-\omega)^n$$

Taking conjugate, we get

$$A+B\omega^2+C\omega = (1-\omega^2)^n$$

$$(x-\omega)(x-\omega^2) = \frac{x^3-1}{x-1} = x^2+x+1.$$

So, $A^2 + B^2 + C^2 - AB - BC - CA$

$$= (A + B\omega + C\omega^2)(A + B\omega^2 + C\omega)$$

$$= (1-\omega)^n (1-\omega^2)^n$$

$$= (1+1+1^2)^n = 3^n.$$

Note that, $A + B + C = 0$.

$$\therefore\ C = -(A+B).$$

$$3^n = A^2 + B^2 + C^2 - AB - BC - CA$$

$$= A^2 + B^2 + (A+B)^2 - AB + (B+A)^2$$

$$= 3(A^2 + AB + B^2).$$

So, $A^2 + B^2 + AB = 3^{n-1}$.

$\square$

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④ Let γ be a root of the

eqn $$x^n + ax + p = 0.$$

Claim: $|\gamma| > 1$.

Proof Let, if possible, $|\gamma| \le 1$.

Then, $$\gamma^n + a\gamma + p = 0$$

$$\Rightarrow p = |a\gamma + \gamma^n|$$
$$= |\gamma| \cdot |a + \gamma^{n-1}|$$
$$\le |a + \gamma^{n-1}|$$
$$\le |a| + |\gamma|^{n-1}$$
$$\le |a| + 1.$$

But, $p > |a| + 1$ is given. $\Rightarrow\Leftarrow$

So we must have $|\gamma| > 1$.

Let, if possible,

$$f(x) = g(x)\,h(x)$$

where g, h are non-constant polynomials with integer coeffts.

Then

$$p = f(0) = \underbrace{g(0)}\,\underbrace{h(0)}$$

$\uparrow$ prime $\qquad$ integers

$\Rightarrow$ at least one among $g(0)$ and $h(0)$ should be ± 1, while the other one should be $\pm p$.

W.l.o.g, say $g(0) = \pm 1$.

Suppose the zeros of $g(x)$ as $r_1, r_2, \ldots, r_k$ $(k \geq 1)$.

Since $r_1, r_2, \ldots, r_k$ are also the roots of $f(x)$,

$$|r_j| > 1 \text{ for each } j.$$

Since, $\underline{g(x)}\,\underline{h(x)} = x^n + ax + p$,

the leading coeff of g can be ± 1.

($\because$ g, h both have integral coeffs)

$$g(x) = \underbrace{\boxed{}}_{\pm 1} (x - r_1) \cdots (x - r_k)$$

Putting $x = 0$, and taking modulus.

$$|g(0)| = \underline{|r_1|} \times \underline{|r_2|} \times \cdots \times \underline{\underline{|r_k|}} > 1.$$

But $|g(0)| = 1$. Thus, we get a contradiction, which completes the proof. $\square$