Ramanujan School of Mathematics

Class Test on Complex Numbers

July 3, 2021

Time allotted: 1 hour | Total marks: $10 \times 2 = 20$

Attempt any TWO questions.

Show all your rough work – partial solutions may be rewarded. You may use any theorem/result without proving it again; but you have to state it properly.

1. Given a triangle ABC, construct two squares $ABMN$ and $ACPQ$ outwardly (such that the squares do not have any overlap with $\triangle ABC$). Let $AD \perp BC$ with D on BC, and E be the midpoint of NQ. Show that the points D, A, E are collinear.

2. Determine, with proof, the value of
$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}.$$

3. Suppose that $P(x), Q(x), R(x), S(x)$ are polynomials such that
$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$
holds for every $x \in \mathbb{C}$. Prove that $x - 1$ is a factor of $S(x)$.

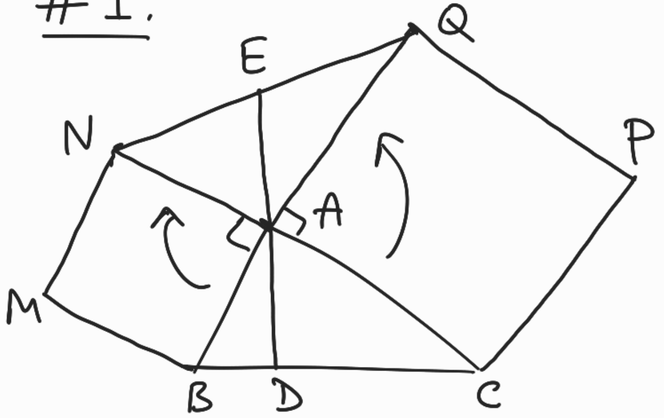
4. Let n be any positive integer. Define
$$A = \binom{n}{0} - \binom{n}{3} + \binom{n}{6} - \cdots, \quad B = -\binom{n}{1} + \binom{n}{4} - \binom{n}{7} + \cdots,$$
$$C = \binom{n}{2} - \binom{n}{5} + \binom{n}{8} - \cdots.$$
Show that, (i) $A^2 + B^2 + C^2 - AB - BC - CA = 3^n$, (ii) $A^2 + AB + B^2 = 3^{n-1}$.

Take the test honestly, do not cheat to yourself. All the best!

Submit your scanned answers to aditya.online.teaching@gmail.com.

Solutions

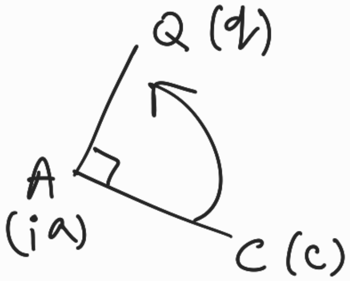
#1.



We setup a complex coord. system such that D is the origin, B and C lie on the real axis and A lies on the imaginary axis.

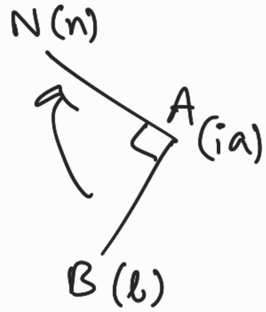
$$B = b, \quad C = c, \quad A = ia \qquad (a, b, c \in \mathbb{R})$$

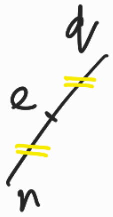
For any other point, say 'X', denote its coord. by 'x'.



$$q - ia = i(c - ia)$$

$$n - ia = -i(b - ia)$$





$$e = \frac{q+n}{2} = i\left(a + \frac{c-b}{2}\right).$$

Since $a + \frac{c-b}{2} \in \mathbb{R}$, this shows that $E(e)$ lies on the imaginary axis, as required.

#2.

Let $z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$.

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \mathrm{Re}(z - z^2 + z^3).$$

$$z - z^2 + z^3 = z(1 - z + z^2) = z\,\frac{z^3+1}{z+1}$$

$$\underbrace{z^7 = -1}_{\text{(by deMoivre's thm)}} \Rightarrow 1/z = -z^6$$

$$= \frac{z^3+1}{1+1/z} = \frac{1+z^3}{1-z^6}$$

$$= \frac{1}{1-z^3}.$$

$w = z^3$ (say). Then $|w| = 1$, so

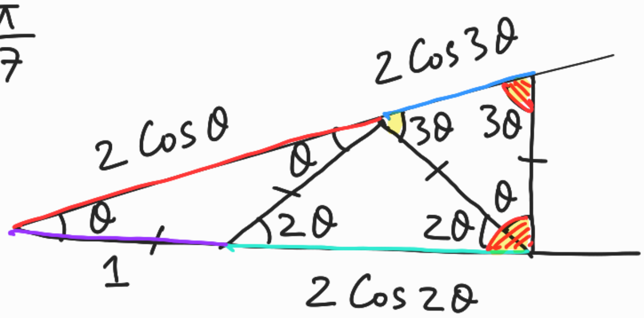
$$\text{Required sum} = \mathrm{Re}\left(\frac{1}{1-w}\right) = \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1-\bar{w}}\right)$$

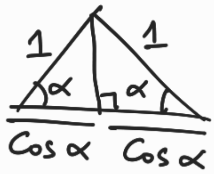
$$= \frac{1}{2}\left(\frac{1}{1-w} + \frac{1}{1-1/w}\right) \quad (\because w\bar{w} = 1)$$

$$= \frac{1}{2}. \quad \text{(Ans)}$$

$\theta = \frac{\pi}{7}$

$\pi - 6\theta$
$= 7\theta - 6\theta$
$= \theta$





$$2\cos\theta + 2\cos 3\theta = 1 + 2\cos 2\theta$$

$$\therefore \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

#3.

$$P(x^5) + x\,Q(x^5) + x^2 R(x^5) = (x^4 + x^3 + x^2 + x + 1)\, S(x) \qquad (*)$$

[If $(*)$ holds for infinitely many values of x then, for LHS − RHS being a poly, we can conclude that LHS − RHS is the zero poly., and therefore $(*)$ also holds for every $x \in \mathbb{C}$.]

$$P(x^5) + x\,Q(x^5) + x^2 R(x^5) = \underline{(x^4 + x^3 + \dots + 1)}\, S(x). \qquad (*)$$

Roots of $x^4 + x^3 + x^2 + x + 1 = 0$ are $\alpha, \alpha^2, \alpha^3, \alpha^4$ where $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Putting these as values of x in $(*)$, we get $(\because \alpha^5 = 1)$

$$P(1) + \alpha^k Q(1) + (\alpha^k)^2 R(1) = 0, \quad k = 1, 2, 3, 4.$$

Now consider the polynomial

$$f(x) = P(1) + x\,Q(1) + x^2 R(1).$$

Note, $\deg f \leq 2$, but f has 4 distinct complex roots, namely $\alpha, \alpha^2, \alpha^3$ and α^4. So, f must be the zero polynomial. Therefore,

$$P(1) = Q(1) = R(1) = 0.$$

Now put $x = 1$ in $(*)$ to get $S(1) = 0$. This gives us the desired conclusion, in view of the factor theorem $\left[S(1) = 0 \Rightarrow x - 1 \mid S(x)\right]$. $\square$

#4. We know,

$$A^2+B^2+C^2-AB-BC-CA$$
$$=(A+B\omega+C\omega^2)(A+B\omega^2+C\omega)$$

where ω is one of the non-real cube roots of unity.

$\omega^2+\omega+1=0$, $\omega^3=1$, $\omega^4=\omega$ etc.

$\omega^{3k}=1$, $\omega^{3k+1}=\omega$, $\omega^{3k+2}=\omega^2$.

Now observe that,

$$A+B\omega+C\omega^2$$
$$=\binom{n}{0}-\binom{n}{1}\omega+\binom{n}{2}\omega^2-\binom{n}{3}+\binom{n}{4}\omega-\binom{n}{5}\omega^2+\dots$$
$$=\binom{n}{0}-\binom{n}{1}\omega+\binom{n}{2}\omega^2-\binom{n}{3}\omega^3+\binom{n}{4}\omega^4-\binom{n}{5}\omega^5+\dots$$
$$=\binom{n}{0}+\binom{n}{1}(-\omega)+\binom{n}{2}(-\omega)^2+\binom{n}{3}(-\omega)^3+\dots$$
$=(1-\omega)^n$, by the Binomial theorem.

$$A+B\omega+C\omega^2=(1-\omega)^n$$

Taking conjugate of both sides,

$$A+B\omega^2+C\omega=(1-\omega^2)^n \quad (\because \omega^2=\overline{\omega},\ \omega=\overline{\omega^2})$$

(Since A, B, C are real).

ω, ω^2 are the roots of $x^2 + x + 1 = 0$, so

$$(x - \omega)(x - \omega^2) = x^2 + x + 1.$$

Hence

$$A^2 + B^2 + C^2 - AB - BC - CA$$

$$= (A + B\omega + C\omega^2)(A + B\omega^2 + C\omega)$$

$$= \left((1 - \omega)(1 - \omega^2)\right)^n$$

$$= (1^2 + 1 + 1)^n = 3^n. \qquad \text{(Proved)}$$

For the second part, note that

$$A + B + C = (1 - 1)^n = 0.$$

So, $C = -(A + B)$. Hence

$$3^n = A^2 + B^2 + C^2 - AB - C(A + B)$$

$$= A^2 + B^2 + (A + B)^2 - AB + (A + B)^2$$

$$= 3(A^2 + B^2 + AB).$$

Therefore, $A^2 + B^2 + AB = 3^{n-1}$. (Proved)