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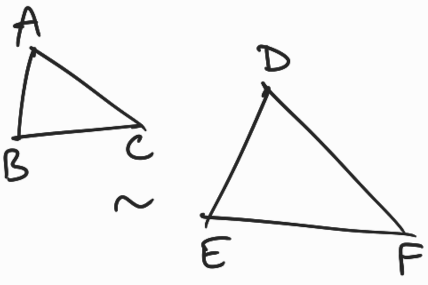


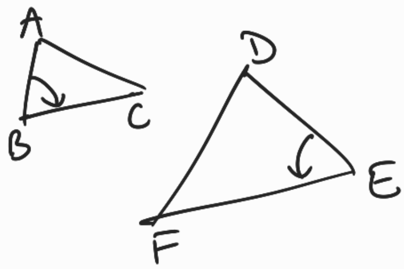
(Same orientation)

$\triangle ABC$ and $\triangle DEF$ are similar to each other if <u>any</u> of the following holds:

(1) $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$

(2) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

(3) $\angle ABC = \angle DEF$, $\dfrac{AB}{BC} = \dfrac{DE}{EF}$.

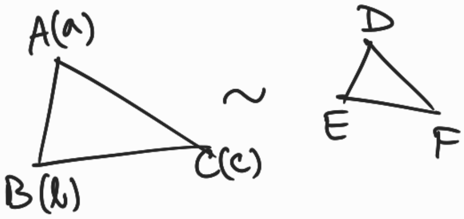


$\angle ABC \neq \angle DEF$

$\angle ABC = \angle FED$

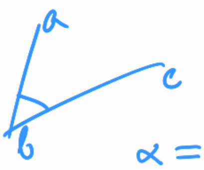
Here $\triangle ABC$ and $\triangle DEF$ have opposite orientation.

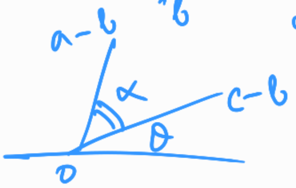
Now let us move to complex no.s. For any point X denote by the lower case x its complex coordinate.



$\overset{(3)}{\iff} \dfrac{AB}{BC} = \dfrac{DE}{EF}$, $\angle ABC = \angle DEF$

$\iff \dfrac{|a-b|}{|b-c|} = \dfrac{|d-e|}{|e-f|}$ and



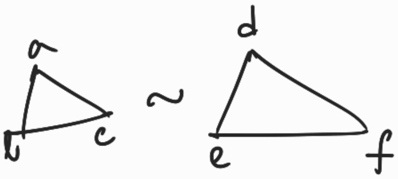


$\alpha = \arg(a-b) - \arg(c-b) = \arg\left(\dfrac{a-b}{b-c}\right) = \arg\left(\dfrac{d-e}{e-f}\right)$ $\Bigg\}$ (*)

$\left(\arg z_1 = \arg z_2 + \arg(z_1/z_2) \quad [\because z_1 = z_2 \times z_1/z_2]\right)$

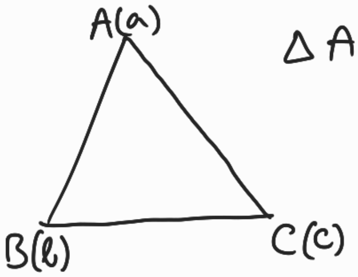
(*) is equivalent to the following: $\dfrac{a-b}{b-c} = \dfrac{d-e}{e-f}$.

<u>Condn for similarity</u>



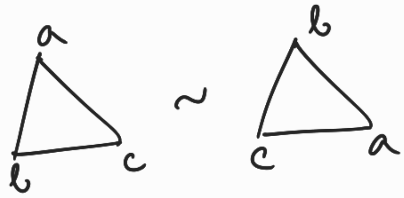
<u>iff</u> $\dfrac{a-b}{b-c} = \dfrac{d-e}{e-f}$.

Applⁿ: Equilateral triangles



ΔABC is equilateral iff

$$\Delta ABC \sim \Delta BCA$$
$$\Updownarrow$$
$$\frac{a-b}{b-c} = \frac{b-c}{c-a}$$
$$\Updownarrow$$
$$a^2+b^2+c^2 = ab+bc+ca.$$



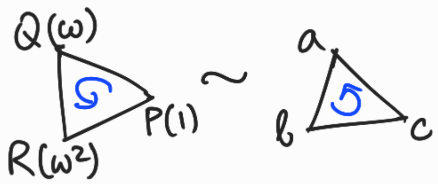
Fact The triangle with vertices $a, b, c \in \mathbb{C}$ is equilateral iff any of the following holds:

1. $|a-b| = |b-c| = |c-a|$
2. $\dfrac{b-a}{c-a} = \dfrac{c-b}{a-b}$ $\qquad$ 2'. $\det\begin{pmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a\end{pmatrix} = 0$
3. $a^2+b^2+c^2 = ab+bc+ca$
4. $a\bar{b} = b\bar{c} = c\bar{a}$
5. $a^2 = bc,\ b^2 = ca,\ c^2 = ab$ $\qquad \omega = \dfrac{-1 \pm i\sqrt{3}}{2}.$
6. $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = 0.$

When b, c, a & $1, \omega, \omega^2$ have same orientation $\qquad$ $=0$ when opposite orientation

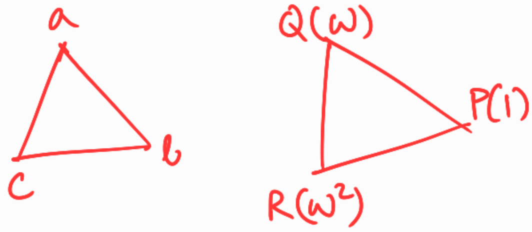
$$\frac{b-c}{c-a} = \frac{1-\omega}{\omega-\omega^2} = \frac{1}{\omega} \implies c-a = b\omega - c\omega$$

ΔBCA
$\sim \Delta PQR$



$$\implies a + b\omega + c\omega^2 = 0.$$
$$(\omega^2 = -1-\omega)$$

Correction: (4) <=> (5) => (3) <=> (1), but (1) need not imply (4) or (5). In words, (4) and (5) are equivalent to each other, but they only imply that the triangle is equilateral, not the other way around.



$$\Delta BCA \sim \Delta PRQ$$

$$\frac{b-c}{c-a} = \frac{1-\omega^2}{\omega^2-\omega} = -\frac{1+\omega}{\omega}$$

$$= -\frac{(-\omega^2)}{\omega}$$

$$= \omega$$

$$\omega^2(b-c) = c-a$$

$$\Longleftrightarrow \quad a + b\omega^2 + c\omega = 0.$$

$$\circledast \quad (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$= a^2 + b^2 + c^2 - ab - bc - ca.$$

This proves ③ $\Longleftrightarrow$ ⑥.

Q. $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the poly. eqn

$$x^4 + ax^3 + bx^2 + ax + c = 0.$$

Show that

$$\underbrace{(\alpha_1^2+1)(\alpha_2^2+1)(\alpha_3^2+1)(\alpha_4^2+1)}_{\prod_{j=1}^{4}(\alpha_j^2+1)} = (1-b+c)^2.$$

$\sum_{j=1}^{4} \alpha_j^2 + 1$

$\prod_{j=1}^{4}(\alpha_j^2+1)$ (just a notation)

$$x^2 + 1 = (x+i)(x-i).$$

So,
$$\prod_{j=1}^{4}(\alpha_j^2+1) = \prod_{j=1}^{4}(\alpha_j+i)(\alpha_j-i)$$

$$= \underbrace{\prod_{j=1}^{4}(\alpha_j-i)}_{A\text{ (say)}} \; \underbrace{\prod_{j=1}^{4}(\alpha_j+i)}_{B\text{ (say)}}$$

Since $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the ~~roots~~ zeros of the poly.

$$P(x) = x^4 + ax^3 + bx^2 + ax + c,$$

we know that

$$P(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

$$= (\alpha_1 - x)(\alpha_2 - x)(\alpha_3 - x)(\alpha_4 - x).$$

Hence, $A = P(i)$, $B = P(-i)$.

$$P(i) = i^4 + \cancel{ai^3} + bi^2 + \cancel{ai} + c = 1 - b + c.$$

Similarly, $P(-i) = 1 - b + c$. So the desired quantity equals $(1-b+c)^2$.

Q. Suppose that A, B, C are angles satisfying

$$\cos 2A + \cos 2B + \cos 2C = \cos(A+B) + \cos(B+C) + \cos(C+A) \quad \text{①}$$

and,

$$\sin 2A + \sin 2B + \sin 2C = \sin(A+B) + \sin(B+C) + \sin(C+A) \quad \text{(II)}$$

Show that, $\cos A + \cos B + \cos C = \sin A + \sin B + \sin C$.

Let $a = \cos A + i \sin A$, $b = \cos B + i \sin B$,

$c = \cos C + i \sin C$.

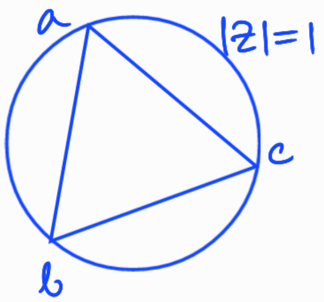
Eqn ① $+ \; i \times$ Eqn (II) gives the following

$$\sum_{cyc} (\cos 2A + i \sin 2A)$$
$$= \sum_{cyc} (\cos (A+B) + i \sin (A+B))$$

Which is same as saying

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

This tells us that a, b, c are vertices of an equilateral triangle. Moreover, $|a| = |b| = |c| = 1$.



$\therefore$ The circumcircle of $\triangle ABC$ is the unit circle centred at the origin.

Since $\triangle ABC$ is equilateral, its centroid is same as its circumcentre, which is the origin.

So, $\dfrac{a+b+c}{3} = 0$

$\Rightarrow (\cos A + \cos B + \cos C) + i(\sin A + \sin B + \sin C) = 0$

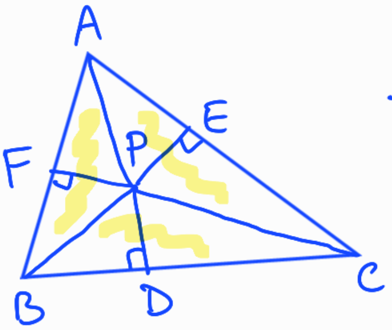
$\Rightarrow \cos A + \cos B + \cos C = \sin A + \sin B + \sin C = 0.$

Hence proved.

Q. P be a varying point inside $\triangle ABC$. Let D, E, F be the feet of the altitudes from P onto the sides BC, CA, AB, respectively.

For which P is the following sum minimized?

$$S = \frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}.$$



$$\tfrac{1}{2} PD \cdot BC + \tfrac{1}{2} PE \cdot CA + \tfrac{1}{2} PF \cdot AB$$
$$= \text{Area}(ABC)$$
$$= \Delta \text{ (say)}$$

$$(*) \quad \boxed{\left(\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}\right)\big(\underbrace{PD \cdot BC + PE \cdot CA + PF \cdot AB}_{=2\Delta}\big) \geq (BC + CA + AB)^2}$$

(by Cauchy-Schwarz inequality)

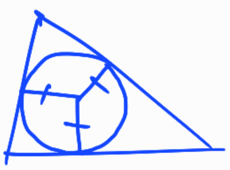
Recall

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq (ax + by + cz)^2$$

with equality iff $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$.

$$\boxed{\text{So, equality holds in } (*) \text{ iff } \frac{1}{PD} = \frac{1}{PE} = \frac{1}{PF}}$$

$\Updownarrow$



P is equidistant from the 3 sides

$\iff$ P is the incentre of ΔABC

Thus, $(*)$ gives

$$\boxed{S = \frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \geq \frac{(2s)^2}{2\Delta}} \to \text{fixed quantity (indept. of } P)$$

and S attains this min value iff P = Incentre of ΔABC. (Ans)